Lattices in potentially semi-stable representations and weak $(\varphi, \hat{G})$-modules

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Abstract

Let p be a prime number and $r \geq 0$ an integer. In this paper, we prove that there exists an anti-equivalence between the category of weak $(\varphi, \hat{G})$-modules of height $\leq r$ and a certain subcategory of the category of Galois stable $\mathbb{Z}_p$-lattices in potentially semi-stable representations with Hodge-Tate weights in $[0, r]$. This gives an answer to a Tong Liu's question about the essential image of a functor on weak $(\varphi, \hat{G})$-modules. For a proof, following Liu's methods, we construct linear algebraic data which classifies lattices in potentially semi-stable representations.

1 Introduction

Let K be a complete discrete valuation field of mixed characteristics $(0, p)$ with perfect residue field. We take a system of p-power roots $(\pi_n)_{n \geq 0}$ of a uniformizer π of K such that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. We denote by G_K and G_{K_n} absolute Galois groups of K and $K_n := K(\pi_n)$, respectively.

For applications to interesting problems such as modularity liftings, it is useful to study an integral version of Fontaine's p-adic Hodge theory, which is called integral p-adic Hodge theory. It is important in integral p-adic Hodge theory to construct "good" linear algebraic data which classify G_K-stable $\mathbb{Z}_p$-lattices in semi-stable, or crystalline, $\mathbb{Q}_p$-representations of G_K with Hodge-Tate weights in $[0, r]$. Nowadays various such linear algebraic data are constructed; for example, so called Fontaine-Laffaille modules, Wach modules and Breuil modules. It is one of the obstructions for the use of these algebraic data that we can not use them without restrictions on the absolute ramification index e of K and (or) r. In [Li3], based on a Kisin's insight [Ki] for a classification of lattices in semi-stable representations, Tong Liu defined notions of $(\varphi, \hat{G})$-modules and weak $(\varphi, \hat{G})$-modules. He constructed a contravariant fully faithful functor $\hat{T}$ from the category of weak $(\varphi, \hat{G})$-modules of height $\leq r$ into the the category of free $\mathbb{Z}_p$-representations of G_K. It is the main theorem of *loc. cit.* that, without any restriction on e and r, $\hat{T}$ induces an anti-equivalence between the category of $(\varphi, \hat{G})$-modules of height $\leq r$ and the category of lattices in semi-stable $\mathbb{Q}_p$-representations of G_K with Hodge-Tate weights in $[0, r]$. In the end of *loc. cit.*, he posed the following question:

Question 1.1. What is the essential image of the functor $\hat{T}$ on *weak* $(\varphi, \hat{G})$-modules?

He showed that, if a representation of G_K corresponds to a weak $(\varphi, \hat{G})$-module of height $\leq r$, then it is semi-stable over K_n for some $n \geq 0$ and has Hodge-Tate weights in $[0, r]$. However, the converse does not hold in general.

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In this paper, we give an answer to Question 1.1. Denote by m_0 the maximum integer such that K contains p^{m_0}-th roots of unity. For any non-negative integer n, we denote by $\mathcal{C}^r_n$ the category of free $\mathbb{Z}_p$-representations T of G_K which satisfy the following property; there exists a semi-stable $\mathbb{Q}_p$-representation V of G_K with Hodge-Tate weights in $[0,r]$ such that $T\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is isomorphic to V as representations of G_{K_n}. Our main result is as follows.

Theorem 1.2. *The essential image of the functor $\hat{T}$ is $\mathcal{C}^r_{m_0}$.*

Therefore, we conclude that $\hat{T}$ induces an anti-equivalence between the category of weak $(\varphi,\hat{G})$-modules of height $\leq r$ and the category $\mathcal{C}^r_{m_0}$.

The crucial part of our proof is to show the relation

$$\mathcal{C}^r_{m_0}\subset\mathcal{C}^r\subset\mathcal{C}^r_m$$

where $\mathcal{C}^r$ is the essential image of the functor $\hat{T}$ and m is the maximum integer such that the maximal unramified extension of K contains p^m-th roots of unity (cf. Lemma 3.1). We have two keys for our proof of this statement. The first one is Proposition 3.13, which gives a relation between weak $(\varphi,\hat{G})$-modules and "finite height" representations. For the proof, following the method of Liu's arguments of [Li3] and [Li4], we construct certain linear data which classifies lattices in potentially semi-stable representations. This is a direct generalization of the main result of [Li3] (the idea for our proof is essentially due to Liu's previous works). The second one is Proposition 3.15; it says that the G_{K_n}-action of a finite height representation of G_K which is semi-stable over K_n extends to a G_K-action which is semi-stable over K.

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Notation : For any topological group H, a free $\mathbb{Z}_p$-representation of H (resp. a $\mathbb{Q}_p$-representation of H) is a finitely generated free $\mathbb{Z}_p$-module equipped with a continuous $\mathbb{Z}_p$-linear H-action (resp. a finite dimensional $\mathbb{Q}_p$-vector space equipped with a continuous $\mathbb{Q}_p$-linear H-action). We denote by $\mathrm{Rep}_{\mathbb{Z}_p}(H)$ (resp. $\mathrm{Rep}_{\mathbb{Q}_p}(H)$) the category of them. For any field F, we denote by G_F the absolute Galois group of F (for a fixed separable closure of F).

2 Preliminary

In this section, we recall some results on Liu's $(\varphi,\hat{G})$-modules and related topics. Throughout this paper, let $p\geq 2$ be a prime number. Let K be a complete discrete valuation field of mixed characteristics $(0,p)$ with perfect residue field k. Let L be a finite extension of K. Take a uniformizer π_L of L and a system of p-power roots $(\pi_{L,n})_{n\geq 0}$ of π_L such that $\pi_{L,0}=\pi_L$ and $\pi^p_{L,n+1}=\pi_{L,n}$. We denote by k_L the residue field of L. Put $L_n=L(\pi_{L,n}), L_\infty=\cup_{n\geq 0}L_n$ and define $\hat{L}$ to be the Galois closure of L_∞ over L. We denote by H_L and $\hat{G}_L$ the Galois group of $\hat{L}/L_\infty$ and $\hat{L}/L$, respectively. We denote by K^{ur} and L^{ur} maximal unramified extensions of K and L, respectively. Note that we have $L^{\mathrm{ur}}=LK^{\mathrm{ur}}$.

Let $R=\varprojlim\mathcal{O}_{\overline{K}}/p$, where $\mathcal{O}_{\overline{K}}$ is the integer ring of $\overline{K}$ and the transition maps are given by the p-th power map. We write $\underline{\pi_L}:=(\pi_{L,n})_{n\geq 0}\in R$. Let $\mathfrak{S}_L:=W(k_L)[\![u_L]\!]$ be the formal power series ring with indeterminate u_L. We define a Frobenius endomorphism φ of $\mathfrak{S}_L$ by $u_L\mapsto u_L^p$ extending the Frobenius of $W(k_L)$. The $W(k_L)$-algebra embedding $W(k_L)[u_L]\hookrightarrow W(R)$ defined by $u_L\mapsto[\underline{\pi_L}]$ extends to $\mathfrak{S}_L\hookrightarrow W(R)$ where $[*]$ is the Teichmüller representative.

We denote by $\mathrm{Mod}^r_{/\mathfrak{S}_L}$ the category of φ-modules $\mathfrak{M}$ over $\mathfrak{S}_L$ which satisfy the following:

- $\mathfrak{M}$ is free of finite type over $\mathfrak{S}_L$; and
- $\mathfrak{M}$ is of height $\leq r$ in the sense that $\mathrm{coker}(1\otimes\varphi\colon\mathfrak{S}_L\otimes_{\varphi,\mathfrak{S}_L}\mathfrak{M}\to\mathfrak{M})$ is killed by $E_L(u_L)^r$.

Here, $E_L(u_L)$ is the minimal polynomial of π_L over $W(k_L)[1/p]$, which is an Eisenstein polynomial. We call objects of this category *Kisin modules of height* $\leq r$ *over* $\mathfrak{S}_L$. We define a contravariant functor $T_{\mathfrak{S}_L}\colon \mathrm{Mod}^r_{/\mathfrak{S}_L} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_{L_\infty})$ by

$$T_{\mathfrak{S}_L}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S}_L,\varphi}(\mathfrak{M}, W(R))$$

for an object $\mathfrak{M}$ of $\mathrm{Mod}^r_{/\mathfrak{S}_L}$. Here a G_{L_∞}-action on $T_{\mathfrak{S}_L}(\mathfrak{M})$ is given by $(\sigma.g)(x) = \sigma(g(x))$ for $\sigma \in G_{L_\infty}, g \in T_{\mathfrak{S}_L}(\mathfrak{M}), x \in \mathfrak{M}$.

Proposition 2.1 ([Ki, Corollary 2.1.4 and Proposition 2.1.12])**.** *The functor* $T_{\mathfrak{S}_L}\colon \mathrm{Mod}^r_{/\mathfrak{S}_L} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_{L_\infty})$ *is exact and fully faithful.*

Let S_L be the p-adic completion of $W(k_L)[u_L, \frac{E_L(u_L)^i}{i!}]_{i\geq 0}$ and endow S_L with the following structures:

- a continuous $\varphi_{W(k_L)}$-semilinear Frobenius $\varphi\colon S_L \to S_L$ defined by $u_L \mapsto u_L^p$.
- a continuous $W(k_L)$-linear derivation $N\colon S_L \to S_L$ defined by $N(u_L) = -u_L$.
- a decreasing filtration $(\mathrm{Fil}^i S_L)_{i\geq 0}$ on S_L. Here $\mathrm{Fil}^i S_L$ is the p-adic closure of the ideal generated by $\frac{E_L(u_L)^j}{j!}$ for all $j \geq i$.

The embedding $\mathfrak{S}_L \hookrightarrow W(R)$ defined above extends to $\mathfrak{S}_L \hookrightarrow S_L \hookrightarrow A_{\mathrm{cris}}$ and $S_L[1/p] \hookrightarrow B^+_{\mathrm{cris}}$. We take a primitive p-power root ζ_{p^n} of unity for $n \geq 0$ such that $\zeta^p_{p^{n+1}} = \zeta_{p^n}$. We set $\underline{\varepsilon} := (\zeta_{p^n})_{n\geq 0} \in R$ and $t := -\log([\underline{\varepsilon}]) \in A_{\mathrm{cris}}$. For any integer $n \geq 0$, let $t^{\{n\}} := t^{r(n)}\gamma_{\tilde{q}(n)}(\frac{t^{p-1}}{p})$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $\tilde{q}(n) \geq 0,\ 0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ the standard divided power. Now we denote by $\nu\colon W(R) \to W(\overline{k})$ a unique lift of the projection $R \to \overline{k}$, which extends to a map $\nu\colon B^+_{\mathrm{cris}} \to W(\overline{k})[1/p]$. For any subring $A \subset B^+_{\mathrm{cris}}$, we put $I_+A = \mathrm{Ker}(\nu \text{ on } B^+_{\mathrm{cris}}) \cap A$.

We define a subring $\mathcal{R}_L$, containing S_L, of B^+_{cris} as below:

$$\mathcal{R}_L := \left\{ \sum_{i=0}^{\infty} f_i t^{\{i\}} \mid f_i \in S_L[1/p] \text{ and } f_i \to 0 \text{ as } i \to \infty \right\}.$$

Furthermore, we define $\widehat{\mathcal{R}}_L := \mathcal{R}_L \cap W(R)$. We see that S_L is not G_L-stable under the action of G_L in B^+_{cris}. However, $\mathcal{R}_L, \widehat{\mathcal{R}}_L, I_+\mathcal{R}_L$ and $I_+\widehat{\mathcal{R}}_L$ are G_L-stable. Furthermore, they are stable under Frobenius in B^+_{cris}. By definition G_L-actions on them factor through $\hat{G}_L$.

For an object $\mathfrak{M}$ of $\mathrm{Mod}^r_{/\mathfrak{S}_L}$, the map $\mathfrak{M} \to \widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$ defined by $x \mapsto 1 \otimes x$ is injective. By this injection, we often regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S}_L)$-stable submodule of $\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$.

Definition 2.2. A *weak* $(\varphi, \hat{G}_L)$*-module of height* $\leq r$ *over* $\mathfrak{S}_L$ is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G}_L)$ where

(1) $(\mathfrak{M}, \varphi)$ is an object of $\mathrm{Mod}^r_{/\mathfrak{S}_L}$,

(2) $\hat{G}_L$ is an $\widehat{\mathcal{R}}_L$-semilinear continuous $\hat{G}_L$-action on $\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$,

(3) the $\hat{G}_L$-action commutes with $\varphi_{\widehat{\mathcal{R}}_L} \otimes \varphi_{\mathfrak{M}}$, and

(4) $\mathfrak{M} \subset (\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M})^{H_L}$.

Furthermore, we say that $\hat{\mathfrak{M}}$ is a $(\varphi, \hat{G}_L)$*-module of height* $\leq r$ *over* $\mathfrak{S}_L$ if $\hat{\mathfrak{M}}$ satisfies the additional condition;

(5) $\hat{G}_L$ acts on $\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}/I_+\widehat{\mathcal{R}}_L(\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M})$ trivially.

We always regard $\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$ as a G_L-module via the projection $G_L \twoheadrightarrow \hat{G}_L$. We denote by ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}$ (resp. $\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}$) the category of weak $(\varphi,\hat{G}_L)$-modules of height $\leq r$ over $\mathfrak{S}_L$ (resp. the category of $(\varphi,\hat{G}_L)$-modules of height $\leq r$ over $\mathfrak{S}_L$).

We define a contravariant functor $\hat{T}_L\colon {}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_L)$ by

$$\hat{T}_L(\hat{\mathfrak{M}}) = \mathrm{Hom}_{\widehat{\mathcal{R}}_L,\varphi}(\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}, W(R))$$

for an object $\hat{\mathfrak{M}} = (\mathfrak{M},\varphi,\hat{G}_L)$ of ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}$. Here a G_L-action on $\hat{T}_L(\hat{\mathfrak{M}})$ is given by $(\sigma.g)(x) = \sigma(g(\sigma^{-1}x))$ for $\sigma \in G_L, g \in \hat{T}_L(\hat{\mathfrak{M}}), x \in \widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$.

Remark 2.3. We should remark that notations $L_n, \mathfrak{S}_L, \widehat{\mathcal{R}}_L, \mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}, \ldots$ above *depend on the choices of a uniformizer π_L of L and a system $(\pi_{L,n})_{n\geq 0}$ of p-power roots of π_L*. Conversely, if we fix the choice of π_L and $(\pi_{L,n})_{n\geq 0}$, such notations are uniquely determined.

Theorem 2.4. *(1) ([Li3, Theorem 2.3.1 (1)]) Let $\hat{\mathfrak{M}} = (\mathfrak{M},\varphi,\hat{G}_L)$ be an object of ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}$. Then the map*

$$\theta\colon T_{\mathfrak{S}_L}(\mathfrak{M}) \to \hat{T}_L(\hat{\mathfrak{M}})$$

defined by $\theta(f)(a\otimes x) := a\varphi(f(x))$ for $a \in \widehat{\mathcal{R}}_L$ and $x \in \mathfrak{M}$, is an isomorphism of representations of G_{L_∞}.
(2) ([Li3, Theorem 2.3.1(2)]) The contravariant functor $\hat{T}_L$ gives an anti-equivalence between the following categories:

- *The category of $(\varphi,\hat{G}_L)$-modules of height $\leq r$ over $\mathfrak{S}_L$.*
- *The category of G_L-stable $\mathbb{Z}_p$-lattices in semi-stable $\mathbb{Q}_p$-representations with Hodge-Tate weights in $[0,r]$.*

(3) ([Li3, Theorem 4.2.2]) The contravariant functor $\hat{T}_L\colon {}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_L)$ is fully faithful. Furthermore, its essential image is contained in the category of G_L-stable $\mathbb{Z}_p$-lattices in potentially semi-stable $\mathbb{Q}_p$-representations of G_L which are semi-stable over L_n for some $n \geq 0$ and have Hodge-Tate weights in $[0,r]$.

Remark 2.5. Put $m = \max\{i \geq 0; \zeta_{p^i} \in L^{\mathrm{ur}}\}$. We claim that any $\mathbb{Q}_p$-representation of G_L which is semi-stable over L_n for some $n \geq 0$ is always semi-stable over L_m.

In the former half part of the proof of [Li3, Theorem 4.2.2], a proof of this claim with "$m = \max\{i \geq 0; \zeta_{p^i} \in L\}$" is written. Unfortunately, there is a gap in the proof. In the proof, the assumption that the extension $L(\zeta_n,\pi_{L,n})/L$ is totally ramified is implicitly used (p. 133, between *l.* 14 and *l.* 21 of [Li3]). However, this condition is not satisfied in general. So we need a little modification. Put $m = \max\{i \geq 0; \zeta_{p^i} \in L^{\mathrm{ur}}\}$ as the beginning. Denote by $\widehat{L^{\mathrm{ur}}}$ the completion of L^{ur}. We remark that the completion of the maximal unramified extension of L_n is just $\widehat{L^{\mathrm{ur}}}(\pi_{L,n})$. Let V be a $\mathbb{Q}_p$-representation of G_L which is semi-stable over L_n for some $n \geq 0$. Then V is semi-stable over $\widehat{L^{\mathrm{ur}}}(\pi_{L,n})$. We remark that the proof of [Li3, Theorem 4.2.2] exactly holds at least under the assumption that the residue field of the base field is algebraically closed. (We need only the first paragraph of *loc. cit.* here.) Thus we know that V is semi-stable over $\widehat{L^{\mathrm{ur}}}(\pi_{L,m})$ and thus we obtain the claim.

Now we restate Theorem 1.2 with the above setting of notation and give a further result. Fix the choice of a uniformizer π_K of K and a system $(\pi_{K,n})_{n\geq 0}$ of p-power roots of π_K, and define notations $K_n, \mathrm{Mod}^{r,\hat{G}_K}_{/\mathfrak{S}_K}, \ldots$ with respect to them. Recall that m_0 (resp. m) is the maximum integer such that K (resp. K^{ur}) contains p^{m_0}-th (resp. p^m-th) roots of unity. We note that the inequality $m_0 \leq m$ always holds. For any non-negative integer n, we denote by $\mathcal{C}^r_n$ the category

of free $\mathbb{Z}_p$-representations T of G_K which satisfy the following property; there exists a semi-stable $\mathbb{Q}_p$-representation V of G_K with Hodge-Tate weights in $[0,r]$ such that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to V as representations of G_{K_n}.

Our goal in this paper is to show the following:

Theorem 2.6. *The essential image of the functor* $\hat{T}_K \colon {}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}_K}_{/\mathfrak{S}_K} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ *is* $\mathcal{C}^r_{m_0}$.

As an immediate consequence of the above theorem, we obtain

Corollary 2.7. *The functor* $\hat{T}_K$ *induces an anti-equivalence* ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}_K}_{/\mathfrak{S}_K} \xrightarrow{\sim} \mathcal{C}^r_{m_0}$.

For later use, we end this section by describing the following proposition.

Proposition 2.8. *Let L be a finite totally ramified extension of K. Then the restriction functor from the category of semi-stable $\mathbb{Q}_p$-representations of G_K into the category of semi-stable $\mathbb{Q}_p$-representations of G_L is fully faithful.*

Proof. In view of the theory of Fontaine's filtered (φ, N)-modules, the result immediately follows from calculations of elementary linear algebras. □

3 Proof of Main Theorem

Our main goal in this section is to give a proof of Theorem 2.6. In the first three subsections, we prove the following lemma, which plays an important role in our proof.

Lemma 3.1. *Denote by $\mathcal{C}^r$ the essential image of* $\hat{T}_K \colon {}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}_K}_{/\mathfrak{S}_K} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$. *Then we have* $\mathcal{C}^r_{m_0} \subset \mathcal{C}^r \subset \mathcal{C}^r_m$.

Clearly, Theorem 2.6 follows immediately from the lemma if $m_0 = m$. However, the condition $m_0 = m$ is not always satisfied. Before starting a main part of this section, we give some remarks about this condition.

Proposition 3.2. *(1) If k is algebraically closed, then $m_0 = m$.*
(2) If $K(\zeta_{p^{m_0+1}})/K$ is ramified, then $m_0 = m$.
(3) Suppose that $\zeta_p \in K$ (resp. $\zeta_4 \in K$) if p is odd (resp. $p = 2$). Then $\hat{K}$ is totally ramified over K if and only if $m_0 = m$.

Proof. The assertion (1) and (2) is clear. We prove (3). If $m_0 < m$, then $K(\zeta_{p^m})$ is a non-trivial unramified extension of K and thus the extension $\hat{K}/K$ is not totally ramified. Conversely, suppose that $\hat{K}/K$ is not totally ramified. Then there exists an integer $n \geq 0$ such that $K(\zeta_{p^n}, \pi_n)/K$ is not totally ramified. This implies so is $K(\zeta_{p^n}, \pi_n)/K(\pi_n)$. We may suppose $n \geq m$. Since $\mathrm{Gal}(K(\zeta_{p^n}, \pi_n)/K(\pi_n))$ is isomorphic to $\mathbb{Z}/p^{n-m_0}\mathbb{Z}$ (here we need the assumption $\zeta_p \in K$ (resp. $\zeta_4 \in K$) if p is odd (resp. $p = 2$)), any subfield of $K(\zeta_{p^n}, \pi_n)/K(\pi_n)$ is of the form $K(\zeta_{p^l}, \pi_n)$ for $m_0 \leq l \leq n$. Thus there exists an integer $m_0 \leq l_0 \leq n$ such that $K^{\mathrm{ur}}(\pi_n) \cap K(\zeta_{p^n}, \pi_n) = K(\zeta_{p^{l_0}}, \pi_n)$. We have $\zeta_{p^{l_0}} \in K^{\mathrm{ur}}(\pi_n) \cap K^{\mathrm{ur}}(\zeta_{p^n})$. Since $\zeta_p \in K$ (resp. $\zeta_4 \in K$) if p is odd (resp. $p = 2$), we have also $K^{\mathrm{ur}}(\pi_n) \cap K^{\mathrm{ur}}(\zeta_{p^n}) = K^{\mathrm{ur}}$. This implies $l_0 \leq m$. Since the residue field extension corresponding to $K(\zeta_{p^n}, \pi_n)/K(\pi_n)$ is non-trivial, the extension $K(\zeta_{p^{l_0}}, \pi_n)/K(\pi_n)$ is non-trivial extension and thus so is $K(\zeta_{p^m}, \pi_n)/K(\pi_n)$. This implies $1 < [K(\zeta_{p^m}, \pi_n) : K(\pi_n)] = [K(\zeta_{p^m}) : K]$ and hence $m_0 < m$. □

Remark 3.3. The condition $m_0 = m$ is not always satisfied. Here are some examples.
(1) Suppose $p > 2$. Set $\alpha := (2+p)^{1/(p-1)}, \beta := (-p)^{1/(p-1)}$ and $K := \mathbb{Q}_p(\alpha\beta)$. The field K is totally ramified over $\mathbb{Q}_p$ since the minimal polynomial of $\alpha\beta$ over $\mathbb{Q}_p$ is an Eisenstein polynomial $X^{p-1} - (2+p)(-p)$. It is well-known that $\mathbb{Q}_p(\beta) = \mathbb{Q}_p(\zeta_p)$. The extension $K(\zeta_p)/K$ is not totally ramified since so is $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ and $p > 2$ (note that the residue class of α is not contained in $\mathbb{F}_p$).

Now we take any odd prime p such that the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is unramified (e.g., $p = 3, 5, 7, \dots$). Then $K(\zeta_p)/K$ is an unramified extension. This implies that $m_0 = 0 < m$. (Moreover, we see that $m = 1$.)
(2) Suppose $p = 2$ and set $K := \mathbb{Q}_2(\sqrt{-5})$. Then $K(\zeta_4)/K$ is unramified extension of degree 2, and thus $m_0 = 1 < m$. (Moreover, we see that $m = 2$.)
(3) Let K' be a finite extension of $\mathbb{Q}_p$ such that it contains p-th roots of unity and $K'(\zeta_{p^\infty})/K'$ is a totally ramified extension. Let K'' be an unramified $\mathbb{Z}_p$-extension of K'. We denote by $K'_{(n)}$ and $K''_{(n)}$ the unique degree-p^n-subextensions of $K'(\zeta_{p^\infty})/K'$ and K''/K', respectively. Explicitly, the field $K'_{(n)}$ coincides with $K'(\zeta_{p^{m'_0+n}})$ where $m'_0 = \max\{i \geq 0 \mid \zeta_{p^i} \in K'\}$. If we denote by $M_{(n)}$ the composite field of $K'_{(n)}$ and $K''_{(n)}$, then we have isomorphisms

$$\mathrm{Gal}(M_{(n)}/K') \simeq \mathrm{Gal}(K'_{(n)}/K') \times \mathrm{Gal}(K''_{(n)}/K') \simeq \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$$

Let K be the subfield of $M_{(n)}/K'$ which corresponds to the group of diagonal components of $\mathrm{Gal}(M_{(n)}/K') \simeq \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$ via Galois theory. We consider m_0 and m for this K. Since $K \cap K'_{(n)} = K'$, we know $m_0 = m'_0$. On the other hand, since $M_{(n)} = KL_{(n)} = K(\zeta_{p^{m'_0+n}})$ and the extension $M_{(n)}/K$ is unramified, we have $m \geq m'_0 + n = m_0 + n$.

3.1 Lattices in potentially semi-stable representations

In this subsection we define a notion of $(\varphi, \hat{G}_L, K)$-modules which classifies lattices in potentially semi-stable $\mathbb{Q}_p$-representations of G_K which are semi-stable over L.

Definition 3.4. A $(\varphi, \hat{G}_L, K)$-*module of height* $\leq r$ *over* $\mathfrak{S}_L$ is a pair $(\hat{\mathfrak{M}}, G_K)$ where

(1) $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G}_L)$ is an object of $\mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}$,

(2) G_K is a $W(R)$-semilinear continuous G_K-action on $W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$,

(3) the G_K-action commutes with $\varphi_{W(R)} \otimes \varphi_{\mathfrak{M}}$, and

(4) the $W(R)$-semilinear G_L-action on $W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M} (\simeq W(R) \otimes_{\widehat{\mathcal{R}}_L} (\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}))$ induced from the $\hat{G}_L$-structure of $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}_L}_{/\mathfrak{S}_L}$ coincides with the restriction of the G_K-action of (2) to G_L.

If $(\hat{\mathfrak{M}}, G_K)$ is a $(\varphi, \hat{G}_L, K)$-module of height $\leq r$ over $\mathfrak{S}_L$, we often abuse notations by writing $\hat{\mathfrak{M}}$ for $(\hat{\mathfrak{M}}, G_K)$ for simplicity. We denote by $\mathrm{Mod}^{r,\hat{G}_L,K}_{/\mathfrak{S}_L}$ the category of $(\varphi, \hat{G}_L, K)$-modules of height $\leq r$ over $\mathfrak{S}_L$.

We define a contravariant functor $\hat{T}_{L/K} \colon \mathrm{Mod}^{r,\hat{G}_L,K}_{/\mathfrak{S}_L} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ by

$$\hat{T}_{L/K}(\hat{\mathfrak{M}}) = \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}, W(R))$$

for an object $\hat{\mathfrak{M}}$ of $\mathrm{Mod}^{r,\hat{G}_L,K}_{/\mathfrak{S}_L}$ with underlying Kisin module $\mathfrak{M}$. Here a G_K-action on $\hat{T}_{L/K}(\hat{\mathfrak{M}})$ is given by $(\sigma.g)(x) = \sigma(g(\sigma^{-1}x))$ for $\sigma \in G_K, g \in \hat{T}_{L/K}(\hat{\mathfrak{M}}), x \in W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$. Note that we have natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\widehat{\mathcal{R}}_L,\varphi}(\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}, W(R)) &\xrightarrow{\sim} \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\varphi,\widehat{\mathcal{R}}_L} (\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}), W(R)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}, W(R)).\end{aligned}$$

Thus we obtain

$$\eta \colon \hat{T}_L(\hat{\mathfrak{M}}) \xrightarrow{\sim} \hat{T}_{L/K}(\hat{\mathfrak{M}}). \tag{3.1}$$

This is G_L-equivariant by the condition (4) of Definition 3.4. In particular, $\hat{T}_{L/K}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over L by Theorem 2.4 (2).

The goal of the rest of this subsection is to prove the following theorem.

Theorem 3.5. *The contravariant functor $\hat{T}_{L/K}$ induces an anti-equivalence between the following categories:*

- *The category of $(\varphi, \hat{G}_L, K)$-modules of height $\leq r$ over $\mathfrak{S}_L$.*
- *The category of G_K-stable $\mathbb{Z}_p$-lattices in potentially semi-stable $\mathbb{Q}_p$-representations of G_K which are semi-stable over L and have Hodge-Tate weights in $[0, r]$.*

The above theorem follows by essentially the same arguments of Liu ([Li3], [Li4]), but we write a proof here for the sake of completeness. Before a proof, we recall Liu's comparison morphisms between $(\varphi, \hat{G}_L)$-modules and representations associated with them. Furthermore, we define its variant for $(\varphi, \hat{G}_L, K)$-modules.

Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G}_L)$ be a weak $(\varphi, \hat{G}_L)$-module of height $\leq r$ over $\mathfrak{S}_L$. By identifying $\hat{T}_L(\hat{\mathfrak{M}})$ with $\mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\varphi, \widehat{\mathcal{R}}_L} (\widehat{\mathcal{R}}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}), W(R))$, we define a $W(R)$-linear map

$$\hat{\imath}_L \colon W(R) \otimes_{\widehat{\mathcal{R}}_L} (\widehat{\mathcal{R}}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}) \to W(R) \otimes_{\mathbb{Z}_p} \hat{T}_L^{\vee}(\hat{\mathfrak{M}})$$

by the composite $W(R) \otimes_{\widehat{\mathcal{R}}_L} (\widehat{\mathcal{R}}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}) \to \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}_L(\hat{\mathfrak{M}}), W(R)) \xrightarrow{\sim} W(R) \otimes_{\mathbb{Z}_p} \hat{T}_L^{\vee}(\hat{\mathfrak{M}})$. Here, the first arrow is defined by $x \mapsto (f \mapsto f(x), \forall f \in \hat{T}_L(\hat{\mathfrak{M}}))$ and the second is a natural isomorphism. Also, for a $(\varphi, \hat{G}_L, K)$-module $\hat{\mathfrak{M}}$ of height $\leq r$ over $\mathfrak{S}_L$, we define a natural $W(R)$-linear map

$$\hat{\imath}_{L/K} \colon W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M} \hookrightarrow W(R) \otimes_{\mathbb{Z}_p} \hat{T}_{L/K}^{\vee}(\hat{\mathfrak{M}})$$

by a similar way. Let $\mathfrak{t}$ be an element of $W(R) \setminus pW(R)$ such that $\varphi(\mathfrak{t}) = pE_L(u_L)E_L(0)^{-1}\mathfrak{t}$. Such $\mathfrak{t}$ is unique up to units of $\mathbb{Z}_p$.

Proposition 3.6. *(1) ([Li3, Proposition 3.1.3]) The map $\hat{\imath}_L$ as above is injective, which preserves Frobenius and G_L-actions. Furthermore, we have $\varphi(\mathfrak{t})^r(W(R) \otimes_{\mathbb{Z}_p} \hat{T}_L^{\vee}(\hat{\mathfrak{M}})) \subset \mathrm{Im}\ \hat{\imath}_L$.*
(2) The map $\hat{\imath}_{L/K}$ as above is injective, which preserves Frobenius and G_K-actions. Furthermore, we have $\varphi(\mathfrak{t})^r(W(R) \otimes_{\mathbb{Z}_p} \hat{T}_{L/K}^{\vee}(\hat{\mathfrak{M}})) \subset \mathrm{Im}\ \hat{\imath}_{L/K}$.
(3) Let $\hat{\mathfrak{M}}$ be a $(\varphi, \hat{G}_L, K)$-module of height $\leq r$ over $\mathfrak{S}_L$ with underlying Kisin module $\mathfrak{M}$. Then the following diagram is commutative:

$$\begin{array}{ccc} W(R) \otimes_{\widehat{\mathcal{R}}_L} (\widehat{\mathcal{R}}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}) & \xrightarrow{\hat{\imath}_L} & W(R) \otimes_{\mathbb{Z}_p} \hat{T}_L^{\vee}(\hat{\mathfrak{M}}) \\ \wr \downarrow & & \uparrow \wr\, W(R) \otimes \eta^{\vee} \\ W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M} & \xrightarrow{\hat{\imath}_{L/K}} & W(R) \otimes_{\mathbb{Z}_p} \hat{T}_{L/K}^{\vee}(\hat{\mathfrak{M}}) \end{array}$$

Here, the left vertical arrow is a natural isomorphism and η is defined in (3.1).

Proof. The commutativity of (3) is clear by construction, and the rest assertions follow by essentially the same proof as [Li3, Proposition 3.1.3]. □

In the rest of this subsection, we denote by $\mathrm{Rep}_{\mathbb{Z}_p}^{r, L\text{-st}}(G_K)$ the full subcategory of $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ appeared in Theorem 3.5. The isomorphism η shows below.

Lemma 3.7. *The functor $\hat{T}_{L/K}$ has values in $\mathrm{Rep}_{\mathbb{Z}_p}^{r, L\text{-st}}(G_K)$.*

Next we show the fully faithfulness of the functor $\hat{T}_{L/K}$.

Lemma 3.8. *The functor $\hat{T}_{L/K}$ is fully faithful.*

Proof. Let $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{M}}'$ be $(\varphi, \hat{G}_L, K)$-modules of height $\leq r$ over $\mathfrak{S}_L$ with underlying Kisin modules $\mathfrak{M}$ and $\mathfrak{M}'$, respectively. Take any G_K-equivariant morphism $f\colon \hat{T}_{L/K}(\hat{\mathfrak{M}}) \to \hat{T}_{L/K}(\hat{\mathfrak{M}}')$. By the map η, we identify $\hat{T}_{L/K}(\hat{\mathfrak{M}})$ and $\hat{T}_{L/K}(\hat{\mathfrak{M}}')$ with $\hat{T}_L(\hat{\mathfrak{M}})$ and $\hat{T}_L(\hat{\mathfrak{M}}')$, respectively. Since $\hat{T}_L$ is fully faithful, there exists a unique morphism $\mathfrak{f}\colon \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}}$ of $(\varphi, \hat{G}_L)$-modules of height $\leq r$ over $\mathfrak{S}_L$ such that $\hat{T}_L(\mathfrak{f}) = f$. It is enough to show that $\mathfrak{f}$ is in fact a morphism of $(\varphi, \hat{G}_L, K)$-modules, that is, $W(R) \otimes \mathfrak{f}\colon W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}' \to W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}$ is G_K-equivariant. Consider the following diagram:

$$\begin{array}{ccc} W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M} & \xrightarrow{\hat{\imath}_{L/K}} & W(R) \otimes_{\mathbb{Z}_p} \hat{T}^{\vee}_{L/K}(\hat{\mathfrak{M}}) \\ {\scriptstyle W(R)\otimes\mathfrak{f}}\uparrow & & \uparrow{\scriptstyle W(R)\otimes f^{\vee}} \\ W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}' & \xrightarrow{\hat{\imath}_{L/K}} & W(R) \otimes_{\mathbb{Z}_p} \hat{T}_{L/K}(\hat{\mathfrak{M}}') \end{array}$$

We see that the above diagram is commutative. Since $W(R) \otimes f^{\vee}$ and two horizontal arrows above are G_K-equivariant, so is $W(R) \otimes \mathfrak{f}$. □

Lemma 3.9. *The functor* $\hat{T}_{L/K}\colon \mathrm{Mod}^{r, \hat{G}_L, K}_{/\mathfrak{S}_L} \to \mathrm{Rep}^{r, L\text{-st}}_{\mathbb{Z}_p}(G_K)$ *is essentially surjective if* L *is a Galois extension of* K.

To show this lemma, we recall arguments of [Li4, §2]. Suppose L is a (not necessary totally ramified) Galois extension of K. Let T be an object of $\mathrm{Rep}^{r, L\text{-st}}_{\mathbb{Z}_p}(G_K)$. Put $d = \mathrm{rank}_{\mathbb{Z}_p} T$. Take a $(\varphi, \hat{G}_L)$-module $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G}_L)$ over $\mathfrak{S}_L$ such that $\hat{T}_L(\hat{\mathfrak{M}}) = T|_{G_L}$. We consider the map $\hat{\imath}_L\colon W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M} \hookrightarrow W(R) \otimes_{\mathbb{Z}_p} \hat{T}^{\vee}_L(\hat{\mathfrak{M}}) = W(R) \otimes_{\mathbb{Z}_p} T^{\vee}$. By the same argument as the proof of [Li4, Lemma 2.3.1], we can check the following

Lemma 3.10. $W(R) \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}$ *is stable under the* G_K*-action via* $\hat{\imath}_L$.

We include (a main part of) the proof in *loc. cit.* of this lemma here since we will use this argument again in the next subsection (cf. the proof of Theorem 3.13). By [Br], we know that $\mathcal{D} := S_L[1/p] \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}$ has a structure of a Breuil module[1] which corresponds to $V|_{G_L}$, where $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In particular, we have a monodromy operator $N_{\mathcal{D}}$ on $\mathcal{D}$. Set $D := \mathcal{D}/I_+ S_L[1/p]\mathcal{D}$. There exists a unique φ-compatible $W(k_L)$-linear section $s\colon D \hookrightarrow \mathcal{D}$. Breuil showed in *loc. cit.* that $N_{\mathcal{D}}$ preserves $s(D)$ and thus we can define $\tilde{N} := N_{\mathcal{D}}|_{s(D)}\colon s(D) \to s(D)$. Then the G_L-action on $B^+_{\mathrm{st}} \otimes_{S_L[1/p]} s(D) (= B^+_{\mathrm{st}} \otimes_{\widehat{\mathcal{R}}_L} (\widehat{\mathcal{R}}_L \otimes_{\varphi, \mathfrak{S}_L} \mathfrak{M}))$ induced from the $\hat{G}_L$-structure on $\hat{\mathfrak{M}}$ is given by

$$g(a \otimes x) = \sum_{i=0}^{\infty} g(a)\gamma_i(-\log([\underline{\varepsilon}(g)])) \otimes \tilde{N}^i(x)$$

for any $g \in G_L, a \in B^+_{\mathrm{st}}$ and $x \in s(D)$. Here, $\underline{\varepsilon}(g) := g(\underline{\pi_L})/\underline{\pi_L} \in R^{\times}$. Set

$$\bar{D} := \left\{ \sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes \tilde{N}^i(x) \mid x \in s(D) \right\} \subset B^+_{\mathrm{st}} \otimes_{W(k_L)[1/p]} s(D)$$

where $\mathfrak{u} := \log([\underline{\pi_L}]) \in B^+_{\mathrm{st}}$. This is a φ-stable $W(k_L)[1/p]$-vector space of dimension d. Setting the monodromy $N_{B^+_{\mathrm{st}}}$ on B^+_{st} by $N(\mathfrak{u}) = 1$, we equip $B^+_{\mathrm{st}} \otimes_{W(k_L)[1/p]} s(D)$ (resp. $B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^{\vee}$) with a monodromy operator N by $N := N_{B^+_{\mathrm{st}}} \otimes 1_{s(D)}$ (resp. $N := N_{B^+_{\mathrm{st}}} \otimes 1_{V^{\vee}}$). Then it is easy to see that $\bar{D}$ is stable under N. On the other hand, we have a natural G_K-equivariant injection $\iota\colon B^+_{\mathrm{st}} \otimes_{W(k_L)[1/p]} D_{\mathrm{st}}(V) \hookrightarrow B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^{\vee}$ where $D_{\mathrm{st}}(V) := (B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^{\vee})^{G_L}$ is a filtered (φ, N)-module over L. (Here we remark that $D_{\mathrm{st}}(V)$ is equipped with a natural G_K-action since L/K is Galois.) Since G_L acts on $\bar{D}$ trivially (cf. §7.2 of [Li1]), the image of $\bar{D}$ under the injection

[1] We do not describe the definition of Breuil modules in this note. See §6.1 of [Br] for axioms of Breuil modules.

$B^+_{\mathrm{st}} \otimes_{W(k_L)[1/p]} s(D) = B^+_{\mathrm{st}} \otimes_{\widehat{\mathcal{R}}_L} (\widehat{\mathcal{R}}_L \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}) \overset{\hat{\iota}_{L,B}}{\hookrightarrow} B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^\vee$ is equal to $\iota(D_{\mathrm{st}}(V))$. Here, $\hat{\iota}_{L,B} := B^+_{\mathrm{st}} \otimes \hat{\iota}_L$, which is compatible with Frobenius and monodromy operators. Hence we have an isomorphism $\hat{i}\colon D_{\mathrm{st}}(V) \xrightarrow{\sim} \bar{D}$ which makes the following diagram commutative:

$$\begin{array}{ccc} D_{\mathrm{st}}(V) & \subset & B^+_{\mathrm{st}} \otimes_{W(k_L)[1/p]} D_{\mathrm{st}}(V) \xrightarrow{\quad\iota\quad} B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^\vee \\ {\scriptstyle\wr}\downarrow{\scriptstyle\hat{i}} & & \| \\ \bar{D} & \subset & B^+_{\mathrm{st}} \otimes_{W(k_L)[1/p]} s(D) \xrightarrow{\quad\hat{\iota}_{L,B}\quad} B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^\vee \end{array}$$

Note that $\hat{i}$ is compatible with Frobenius and monodromy operators. We identify $D_{\mathrm{st}}(V)$ with $\bar{D}$ by $\hat{i}$.

Let $e_1, \dots, e_d$ be a $W(k_L)[1/p]$-basis of D, and define a matrix $\bar{N} \in M_d(W(k_L)[1/p])$ by $\tilde{N}(s(e_1), \dots, s(e_d)) = (s(e_1), \dots, s(e_d))\bar{N}$. Put $\bar{e}_j = \sum_{i=0}^{\infty} \gamma_i(\mathfrak{u}) \otimes \tilde{N}^i(s(e_j))$ for any j. Then $\bar{e}_1, \dots, \bar{e}_d$ is a basis of $D_{\mathrm{st}}(V) = \bar{D}$. An easy calculation shows that the monodromy N on $D_{\mathrm{st}}(V) = \bar{D}$ is represented by $\bar{N}$ for this basis, that is, $N(\bar{e}_1, \dots, \bar{e}_d) = (\bar{e}_1, \dots, \bar{e}_d)\bar{N}$. We define a matrix $A_g \in GL_d(W(k_L)[1/p])$ by $g(\bar{e}_1, \dots, \bar{e}_d) = (\bar{e}_1, \dots, \bar{e}_d)A_g$ for any $g \in G_K$. Since the G_K-action on $D_{\mathrm{st}}(V) = \bar{D}$ is compatible with N, we have the relation $A_g g(\bar{N}) = \bar{N} A_g$. Consequently, we have

$$g(s(e_1), \dots, s(e_d)) = (s(e_1), \dots, s(e_d))\exp(-\lambda_g \bar{N})A_g \tag{3.2}$$

in $B^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^\vee$, where $\lambda_g := \log([g(\underline{\pi}_L)/\underline{\pi}_L]) \in B^+_{\mathrm{cris}}$. This implies that $B^+_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M} = B^+_{\mathrm{cris}} \otimes_{W(k_L)[1/p]} s(D)$ is stable under the G_K-action via $\hat{\iota}_{L,B}$. Now Lemma 3.10 follows by an easy combination of Proposition 3.6 (1) and [Li3, Lemma 3.2.2] (cf. the first paragraph of the proof of [Li4, Lemma 2.3.1]).

Proof of Lemma 3.9. We continue to use the same notation as above. By Lemma 3.10, we know that $\hat{\mathfrak{M}}$ has a structure of an object of $\mathrm{Mod}^{r,\hat{G}_L,K}_{/\mathfrak{S}_L}$ with the property that the map $W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M} \overset{\hat{\iota}_L}{\hookrightarrow} W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee_L(\hat{\mathfrak{M}}) = W(R) \otimes_{\mathbb{Z}_p} T^\vee$ is G_K-equivariant. Let $\eta\colon \hat{T}_L(\hat{\mathfrak{M}}) \xrightarrow{\sim} \hat{T}_{L/K}(\hat{\mathfrak{M}})$ be the isomorphism defined in (3.1). By Proposition 3.6 (3), we know that $W(R) \otimes \eta^\vee$ induces an isomorphism $\hat{\iota}_L(W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}) \xrightarrow{\sim} \hat{\iota}_{L/K}(W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M})$, which is G_K-equivariant. Since $\varphi(\mathfrak{t})^r(W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee_L(\hat{\mathfrak{M}}))$ (resp. $\varphi(\mathfrak{t})^r(W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee_{L/K}(\hat{\mathfrak{M}}))$) is contained in $(\hat{\iota}_L(W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}))$ (resp. $(\hat{\iota}_{L/K}(W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}))$), we known that the map $\varphi(\mathfrak{t})^r(W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee_L(\hat{\mathfrak{M}})) \xrightarrow{\sim} \varphi(\mathfrak{t})^r(W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee_{L/K}(\hat{\mathfrak{M}}))$ induced from $W(R) \otimes \eta^\vee$ is G_K-equivariant. Thus so is $\eta\colon T = \hat{T}_L(\hat{\mathfrak{M}}) \xrightarrow{\sim} \hat{T}_{L/K}(\hat{\mathfrak{M}})$. □

Remark 3.11. Let $\hat{e}_1, \dots, \hat{e}_d$ be a $\mathfrak{S}_K$-basis of $\varphi^*\mathfrak{M}$, which is also an $S_K[1/p]$-basis of $\mathcal{D}$. Denote by e_i the image of $\hat{e}_i$ under the projection $\mathcal{D} \twoheadrightarrow D$. Then $e_1, \dots, e_d$ is a $W(k)[1/p]$-basis of D. For these basis, we see that the matrix $A_g \in GL_d(W(k_L)[1/p])$ as above is in fact contained in $GL_d(W(k_L))$ by Lemma 3.9. (However, we never use this fact in the present paper.)

Lemma 3.12. *The functor $\hat{T}_{L/K}\colon \mathrm{Mod}^{r,\hat{G}_L,K}_{/\mathfrak{S}_L} \to \mathrm{Rep}^{r,L\text{-st}}_{\mathbb{Z}_p}(G_K)$ is essentially surjective for any finite extension L of K.*

Proof. Let T be an object of $\mathrm{Rep}^{L\text{-st},r}_{\mathbb{Z}_p}(G_K)$. Let L' be the Galois closure of L over K (and fix the choice of a uniformizer of L' and a system of p-power roots of it; see Remark 2.3). Since we have already shown Theorem 3.5 for $\hat{T}_{L'/K}$, we know that there exists a $(\varphi, \hat{G}_{L'}, K)$-module $\hat{\mathfrak{M}}'$ over $\mathfrak{S}_{L'}$ such that $\hat{T}_{L'/K}(\hat{\mathfrak{M}}') \simeq T$ as representations of G_K. On the other hand, we have a unique $(\varphi, \hat{G}_L)$-module $\hat{\mathfrak{M}}$ such that $T \simeq \hat{T}_L(\hat{\mathfrak{M}})$ as representations of G_L since T is semi-stable over L. We denote by $\mathfrak{M}'$ and $\mathfrak{M}$ underlying Kisin modules of $\hat{\mathfrak{M}}'$ and $\hat{\mathfrak{M}}$, respectively. By [Li5, Theorem 3.2.1] and Proposition 3.6 (3), the image of $W(R) \otimes_{\varphi,\mathfrak{S}_{L'}} \mathfrak{M}'$ under $\hat{\iota}_{L'/K}$ is equal to that of $W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$ under $\hat{\iota}_L$. Hence we have a φ-equivariant isomorphism $W(R) \otimes_{\varphi,\mathfrak{S}_{L'}} \mathfrak{M}' \simeq W(R) \otimes_{\varphi,\mathfrak{S}_L} \mathfrak{M}$. We

define a G_K-action on $W(R)\otimes_{\varphi,\mathfrak{S}_L}\mathfrak{M}$ by this isomorphism. Then $\hat{\mathfrak{M}}$ has a structure of $(\varphi,\hat{G}_L,K)$-module over $\mathfrak{S}_L$ so that $\hat{\imath}_L\colon W(R)\otimes_{\varphi,\mathfrak{S}_L}\mathfrak{M}\hookrightarrow W(R)\otimes_{\mathbb{Z}_p}T^\vee$ is G_K-equivariant. Since $\hat{T}_{L/K}(\hat{\mathfrak{M}})=\mathrm{Hom}_{W(R),\varphi}(W(R)\otimes_{\varphi,\mathfrak{S}_L}\mathfrak{M},W(R))\simeq\mathrm{Hom}_{W(R),\varphi}(W(R)\otimes_{\varphi,\mathfrak{S}_{L'}}\mathfrak{M}',W(R))=\hat{T}_{L'/K}(\hat{\mathfrak{M}}')=T$ as representations of G_K, we have done. □

3.2 $\mathcal{C}^r_{m_0}\subset\mathcal{C}^r$

We prove the relation $\mathcal{C}^r_{m_0}\subset\mathcal{C}^r$ in the assertion of Lemma 3.1. At first, fix the choices of a uniformizer π_K of K and a system $(\pi_{K,n})_{n\geq 0}$ of p-power roots of π_K, and define notations $K_n,\mathfrak{S}_K,\mathrm{Mod}^{r,\hat{G}_K}_{/\mathfrak{S}_K},\dots$ with respect to them (see also Remark 2.3). We also consider notations $\mathfrak{S}_{K_n},S_{K_n},\dots$ with respect to the uniformizer $\pi_{K_n}:=\pi_{K,n}$ of K_n and the system $(\pi_{K,n+m})_{m\geq 0}$ of p-power roots of π_{K_n}. Note that we have $\mathfrak{S}_K\subset\mathfrak{S}_{K_n}$, $S_K\subset S_{K_n}$ and $E_{K_n}(u_{K_n})=E_K(u_K)$ with the relation $u_{K_n}^{p^n}=u_K$.

To show the relation $\mathcal{C}^r_{m_0}\subset\mathcal{C}^r$, it follows from Lemma 2.1.15 of [Ki] that it suffices to show the following.

Proposition 3.13. *Let T be a free $\mathbb{Z}_p$-representation of G_K which is semi-stable over K_n for some $n\leq m_0$ and $T|_{G_{K_\infty}}\simeq T_{\mathfrak{S}_K}(\mathfrak{M})$ for some $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}_K}$. Then there exists a (unique) weak $(\varphi,\hat{G}_K)$-module $\hat{\mathfrak{M}}$ of height $\leq r$ over $\mathfrak{S}_K$ such that $\hat{T}_K(\hat{\mathfrak{M}})\simeq T$.*

Proof. Let T,n and $\mathfrak{M}$ be as in the statement. Note that K_n is a now Galois extension of K for such n, and note also that $\mathfrak{M}_n:=\mathfrak{S}_{K_n}\otimes_{\mathfrak{S}_K}\mathfrak{M}$ is a Kisin module of height $\leq r$ over $\mathfrak{S}_{K_n}$. By Theorem 3.5, there exists a $(\varphi,\hat{G}_{K_n},K)$-module $\hat{\mathfrak{N}}$ over $\mathfrak{S}_{K_n}$ such that $T\simeq\hat{T}_{K_n/K}(\hat{\mathfrak{N}})$. Denote by $\mathfrak{N}$ the underlying Kisin module of $\hat{\mathfrak{N}}$. Since $T_{\mathfrak{S}_{K_n}}(\mathfrak{M}_n)$ is isomorphic to $T_{\mathfrak{S}_{K_n}}(\mathfrak{N})$, we may identify $\mathfrak{N}$ with $\mathfrak{M}_n$. Thus $\mathfrak{M}_n$ is equipped with a structure of a $(\varphi,\hat{G}_{K_n},K)$-module $\hat{\mathfrak{M}}_n$ over $\mathfrak{S}_{K_n}$ such that $T\simeq\hat{T}_{K_n/K}(\hat{\mathfrak{M}}_n)$. Putting $\varphi^*\mathfrak{M}=\mathfrak{S}_K\otimes_{\varphi,\mathfrak{S}_K}\mathfrak{M}$, we know that $G_K(\varphi^*\mathfrak{M})$ is contained in $W(R)\otimes_{\varphi,\mathfrak{S}_{K_n}}\mathfrak{M}_n=W(R)\otimes_{\varphi,\mathfrak{S}_K}\mathfrak{M}$. We claim that $G_K(\varphi^*\mathfrak{M})$ is contained in $\mathcal{R}_K\otimes_{\varphi,\mathfrak{S}_K}\mathfrak{M}$. Admitting this claim, we see that $\mathfrak{M}$ has a structure of a weak $(\varphi,\hat{G}_K)$-module of height $\leq r$ over $\mathfrak{S}_K$ which corresponds to T, and hence we finish a proof.

Put $\mathcal{D}_n=S_{K_n}[1/p]\otimes_{\varphi,\mathfrak{S}_{K_n}}\mathfrak{M}_n$ and $\mathcal{D}=S_K[1/p]\otimes_{\varphi,\mathfrak{S}_K}\mathfrak{M}$. Let $\hat{e}_1,\dots,\hat{e}_d$ be a $\mathfrak{S}_K$-basis of $\varphi^*\mathfrak{M}$, which is an $S_{K_n}[1/p]$-basis of $\mathcal{D}_n$ and an $S_K[1/p]$-basis of $\mathcal{D}$. Denote by e_i the image of $\hat{e}_i$ under the projection $\mathcal{D}\twoheadrightarrow\mathcal{D}/I_+S_K[1/p]=:D$. Then $e_1,\dots,e_d$ is a $W(k)[1/p]$-basis of D. By [Br, Proposition 6.2.1.1], we have a unique φ-compatible section $s\colon D\hookrightarrow\mathcal{D}$ of the projection $\mathcal{D}\twoheadrightarrow D$. Since $\mathcal{D}=S_K[1/p]\otimes_{W(k)[1/p]}s(D)$, there exists a matrix $X\in GL_d(S_K[1/p])$ such that $(\hat{e}_1,\dots,\hat{e}_d)=(s(e_1),\dots,s(e_d))X$. Now we extend the G_K-action on $W(R)\otimes_{\varphi,\mathfrak{S}_{K_n}}\mathfrak{M}_n$ to $B^+_{\mathrm{cris}}\otimes_{W(k)[1/p]}s(D)=B^+_{\mathrm{cris}}\otimes_{W(R)}(W(R)\otimes_{\varphi,\mathfrak{S}_{K_n}}\mathfrak{M}_n)$ by a natural way. Take any $g\in G_K$ and put $\lambda_g=\log([g(\underline{\pi}_{K_n})/\underline{\pi}_{K_n}])$. We see that λ_g is contained in $\mathcal{R}_K$. Recall that K_n is now a totally ramified Galois extension over K. By (3.2), we have $g(s(e_1),\dots,s(e_d))=(s(e_1),\dots,s(e_d))\exp(-\lambda_g\bar{N})A_g$ for some nilpotent matrix $\bar{N}\in M_d(W(k)[1/p])$ and some $A_g\in GL_d(W(k)[1/p])$. Therefore, we obtain $g(\hat{e}_1,\dots,\hat{e}_d)=(\hat{e}_1,\dots,\hat{e}_d)X^{-1}\exp(-\lambda_g\bar{N})A_gg(X)$. Since the matrix $X^{-1}\exp(-\lambda_g\bar{N})A_gg(X)$ has coefficients in $\mathcal{R}_K$, we have done. □

Remark 3.14. We remark that, for any semi-stable $\mathbb{Q}_p$-representation V of G_{K_n} with Hodge-Tate weights in $[0,r]$, there exists a Kisin module $\mathfrak{M}_n\in\mathrm{Mod}^r_{/\mathfrak{S}_{K_n}}$ such that $V|_{G_{K_\infty}}$ is isomorphic to $T_{\mathfrak{S}_{K_n}}(\mathfrak{M}_n)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ (cf. [Ki, Lemma 2.1.15]). The above proposition studies the case where $\mathfrak{M}_n$ descends to a Kisin module over $\mathfrak{S}_K$, but this condition is not always satisfied. An example for this is given in the proof of Proposition 3.22.

3.3 $\mathcal{C}^r\subset\mathcal{C}^r_m$

Next we prove the relation $\mathcal{C}^r\subset\mathcal{C}^r_m$ in the assertion of Lemma 3.1. The key for our proof is the following proposition.

Proposition 3.15. *The restriction functor* $\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Rep}_{\mathbb{Q}_p}(G_{K_n})$ *induces an equivalence between the following categories:*

- *The category of semi-stable* $\mathbb{Q}_p$*-representations of* G_K *with Hodge-Tate weights in* $[0, r]$.
- *The category of semi-stable* $\mathbb{Q}_p$*-representations* V *of* G_{K_n} *with the property that* $V|_{G_{K_\infty}}$ *is isomorphic to* $T_{\mathfrak{S}_K}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *for some* $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_K}$.

The result below immediately follows from the above proposition.

Corollary 3.16. *Let* T *be a free* $\mathbb{Z}_p$*-representation of* G_K *which is semi-stable over* K_n *for some* $n \geq 0$. *Then the following conditions are equivalent:*

- $T|_{G_{K_\infty}}$ *is isomorphic to* $T_{\mathfrak{S}_K}(\mathfrak{M})$ *for some* $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_K}$.
- *There exists a semi-stable* $\mathbb{Q}_p$*-representation* V *of* G_K *with Hodge-Tate weights in* $[0, r]$ *such that* $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is isomorphic to* V *as representations of* $G_{K_{n'}}$ *for some* $n' \geq 0$.

Remark 3.17. In the statement of Corollary 3.16, we can always choose n' to be n. In addition, for a given T, V is uniquely determined up to isomorphism. Furthermore, the association $T \mapsto V$ is functorial. These follow from Proposition 2.8.

Combining this corollary with Theorem 2.4 (3) and Remark 2.5, we obtain the desired relation $\mathcal{C}^r \subset \mathcal{C}^r_m$. Therefore, it suffices to show Proposition 3.15. We begin with the following two lemmas.

Lemma 3.18. *For any* $i \geq 0$, *we have a canonical decomposition*

$$\mathrm{Fil}^i S_{K_n} = \bigoplus_{j=0}^{p^n-1} u_{K_n}^j \mathrm{Fil}^i S_K.$$

Proof. Exercise. □

Lemma 3.19. *Let* $\mathfrak{M}$ *be a Kisin module of height* $\leq r$ *over* $\mathfrak{S}_K$.
(1) $\mathfrak{M}_n := \mathfrak{S}_{K_n} \otimes_{\mathfrak{S}_K} \mathfrak{M}$ *is a Kisin module of height* $\leq r$ *over* $\mathfrak{S}_{K_n}$ *(with Frobenius* $\varphi_{\mathfrak{M}_n} := \varphi_{\mathfrak{S}_{K_n}} \otimes \varphi_{\mathfrak{M}}$*).*
(2) Let $\mathcal{M} := S_K \otimes_{\varphi, \mathfrak{S}_K} \mathfrak{M}$ *and* $\mathcal{M}_n := S_{K_n} \otimes_{\varphi, \mathfrak{S}_K} \mathfrak{M} = S_{K_n} \otimes_{\varphi, \mathfrak{S}_{K_n}} \mathfrak{M}_n$. *Define* $\mathrm{Fil}^i \mathcal{M} := \{x \in \mathcal{M} \mid (1 \otimes \varphi_{\mathfrak{M}})(x) \in \mathrm{Fil}^i S_K \otimes_{\mathfrak{S}_K} \mathfrak{M}\}$ *and* $\mathrm{Fil}^i \mathcal{M}_n := \{x \in \mathcal{M}_n \mid (1 \otimes \varphi_{\mathfrak{M}})(x) \in \mathrm{Fil}^i S_{K_n} \otimes_{\mathfrak{S}_K} \mathfrak{M}\} = \{x \in \mathcal{M}_n \mid (1 \otimes \varphi_{\mathfrak{M}_n})(x) \in \mathrm{Fil}^i S_{K_n} \otimes_{\mathfrak{S}_{K_n}} \mathfrak{M}_n\}$. *Then the natural isomorphism* $S_{K_n} \otimes_{S_K} \mathcal{M} \xrightarrow{\sim} \mathcal{M}_n$ *induces an isomorphism* $S_{K_n} \otimes_{S_K} \mathrm{Fil}^i \mathcal{M} \xrightarrow{\sim} \mathrm{Fil}^i \mathcal{M}_n$.

Proof. The assertion (1) follows immediately by the relation $E_K(u_K) = E_{K_n}(u_{K_n})$. In the rest of this proof we identify $S_{K_n} \otimes_{S_K} \mathcal{M}$ with $\mathcal{M}_n$ by a natural way. We show that $S_{K_n} \otimes_{S_K} \mathrm{Fil}^i \mathcal{M} = \mathrm{Fil}^i \mathcal{M}_n$. The inclusion $S_{K_n} \otimes_{S_K} \mathrm{Fil}^i \mathcal{M} \subset \mathrm{Fil}^i \mathcal{M}_n$ follows from an easy calculation. We have to prove the opposite inclusion. Let $e_1, \ldots, e_d$ be an $\mathfrak{S}_K$-basis of $\mathfrak{M}$ and define a matrix $A \in M_d(\mathfrak{S}_K)$ by $\varphi_{\mathfrak{M}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A$. We put $e_i^* = 1 \otimes e_i \in \varphi^* \mathfrak{M}$ for any i. Then $e_1^*, \ldots, e_d^*$ is an S_{K_n}-basis of $\mathcal{M}_n$. Take $x = \sum_{k=1}^d a_k e_k^* \in \mathrm{Fil}^i \mathcal{M}_n$ with $a_k \in S_{K_n}$. Since $(1 \otimes \varphi_{\mathfrak{M}})(x)$ is contained in $\mathrm{Fil}^i S_{K_n} \otimes_{\mathfrak{S}} \mathfrak{M}$, we see that the matrix

$$X := A \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}$$

has coefficients in $\mathrm{Fil}^i S_{K_n}$. By Lemma 3.18, each a_k can be decomposed as $\sum_{j=0}^{p^n-1} u_{K_n}^j a_k^{(j)}$ for some $a_k^{(j)} \in S_K$. Writing $A = (a_{lk})_{l,k}$ and $X = {}^{\mathrm{t}}(x_1, \ldots, x_d)$, we have

$$x_l = \sum_{k=1}^d a_{lk} a_k = \sum_{j=0}^{p^n-1} u_{K_n}^j \sum_{k=1}^d a_{lk} a_k^{(j)}.$$

By Lemma 3.18 again, we obtain that $\sum_{k=1}^{d} a_{lk}a_k^{(j)} \in \mathrm{Fil}^i S_K$. If we put $x_{(j)} = \sum_{k=1}^{d} a_k^{(j)} e_k^* \in \mathcal{M}$, we have

$$(1 \otimes \varphi_{\mathfrak{M}})(x_{(j)}) = \sum_{l=1}^{d} (\sum_{k=1}^{d} a_{lk}a_k^{(j)}) e_l,$$

which is contained in $\mathrm{Fil}^i S_K \otimes_{\mathfrak{S}_K} \mathfrak{M}$. Therefore, each $x_{(j)}$ is contained in $\mathrm{Fil}^i \mathcal{M}$. Since $x = \sum_{j=0}^{p^n-1} u_{K_n}^j x_{(j)}$, we obtain the fact that x is contained in $S_{K_n} \otimes_{S_K} \mathrm{Fil}^i \mathcal{M}$. □

We proceed a proof of Proposition 3.15. For simplicity, we denote by $\mathbf{R}_1$ (resp. $\mathbf{R}_2$) the former (resp. latter) category appeared in the statement of Proposition 3.15. It is well-known (cf. [Ki, Lemma 2.1.15]) that the essential image of $\mathbf{R}_1$ under the restriction functor $\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Rep}_{\mathbb{Q}_p}(G_{K_n})$ is contained in $\mathbf{R}_2$. Furthermore, the restriction functor $\mathbf{R}_1 \to \mathbf{R}_2$ is fully faithful since K_n is totally ramified over K. Thus it suffices to show the essential surjectivity of the restriction functor $\mathbf{R}_1 \to \mathbf{R}_2$. Let V be a semi-stable $\mathbb{Q}_p$-representations V of G_{K_n} with the property that $V|_{G_{K_\infty}}$ is isomorphic to $T_{\mathfrak{S}_K}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for some $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_K}$. Set $T := T_{\mathfrak{S}_K}(\mathfrak{M})$ and take any G_{K_n}-stable $\mathbb{Z}_p$-lattice T' in V such that $T \subset T'$. There exists a $(\varphi, \hat{G}_{K_n})$-module $\hat{\mathfrak{N}}$ of height $\leq r$ over $\mathfrak{S}_{K_n}$ such that $T' \simeq \hat{T}_{K_n}(\hat{\mathfrak{N}})$. Put $\mathfrak{M}_n = \mathfrak{S}_{K_n} \otimes_{\mathfrak{S}_K} \mathfrak{M}$, which is a Kisin module of height $\leq r$ over $\mathfrak{S}_{K_n}$. Since the functor $T_{\mathfrak{S}_{K_n}}$ from $\mathrm{Mod}^r_{/\mathfrak{S}_{K_n}}$ into $\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_\infty})$ is fully faithful, we obtain a morphism $\mathfrak{N} \to \mathfrak{M}_n$ which corresponds to the inclusion map $T \hookrightarrow T'$. We note that it is injective and its cokernel $\mathfrak{M}_n/\mathfrak{N}$ is killed by a power of p since T'/T is p-power torsion. Set $\mathcal{D}_n := S_{K_n}[1/p] \otimes_{\mathfrak{S}_{K_n}} \mathfrak{N} \simeq S_{K_n}[1/p] \otimes_{\mathfrak{S}_{K_n}} \mathfrak{M}_n$, $\mathcal{D} := S_K[1/p] \otimes_{\mathfrak{S}_K} \mathfrak{M}$, $\mathcal{N} := S_{K_n} \otimes_{\mathfrak{S}_{K_n}} \mathfrak{N}$, $\mathcal{M}_n := S_{K_n} \otimes_{\mathfrak{S}_{K_n}} \mathfrak{M}_n$ and $\mathcal{M} := S_K \otimes_{\mathfrak{S}_K} \mathfrak{M}$. We define filtrations $\mathrm{Fil}^i \mathcal{N}$, $\mathrm{Fil}^i \mathcal{M}_n$ and $\mathrm{Fil}^i \mathcal{M}$ as Lemma 3.19 (2). Note that $\mathcal{D}_n$ has a structure of a Breuil module which corresponds to V. In particular, we have a Frobenius $\varphi_{\mathcal{D}_n}$, a monodromy operator $N_{\mathcal{D}_n}$ and a decreasing filtration $(\mathrm{Fil}^i \mathcal{D}_n)_{i \in \mathbb{Z}}$ on $\mathcal{D}_n$. It is a result of [Br, §6] that we can equip $D := \mathcal{D}_n/I_+ S_{K_n}[1/p]\mathcal{D}_n$ with a structure of filtered (φ, N)-module over K_n which corresponds to V. Now we recall the definition of this structure and also define some additional notations for later use. The Frobenius φ_D and the monodromy N_D on D is defined by $\varphi_D := \varphi_{\mathcal{D}_n} \bmod I_+ S_{K_n}[1/p]\mathcal{D}_n$ and $N_D := N_{\mathcal{D}_n} \bmod I_+ S_{K_n}[1/p]\mathcal{D}_n$. We denote by f_{π_n} and f_π the natural projections $\mathcal{D}_n \twoheadrightarrow \mathcal{D}_n/\mathrm{Fil}^1 S_{K_n}\mathcal{D}_n$ and $\mathcal{D} \twoheadrightarrow \mathcal{D}/\mathrm{Fil}^1 S_K \mathcal{D}$, respectively. There is a unique φ-compatible section $s\colon D \hookrightarrow \mathcal{D}$ of the projection $\mathcal{D} \twoheadrightarrow \mathcal{D}/I_+ S_K[1/p]\mathcal{D} \simeq D$. Note that the composite $D \overset{s}{\hookrightarrow} \mathcal{D} \hookrightarrow \mathcal{D}_n$, which is also denoted by s, is a section of the projection $\mathcal{D}_n \twoheadrightarrow \mathcal{D}_n/I_+ S_{K_n}[1/p]\mathcal{D}_n = D$. Since the composite $D \overset{s}{\to} \mathcal{D}_n \overset{f_{\pi_n}}{\to} \mathcal{D}_n/\mathrm{Fil}^1 S_{K_n}\mathcal{D}_n$ (resp. $D \overset{s}{\to} \mathcal{D} \overset{f_\pi}{\to} \mathcal{D}/\mathrm{Fil}^1 S_K \mathcal{D}$) maps a basis of D to a basis of $\mathcal{D}_n/\mathrm{Fil}^1 S_{K_n}\mathcal{D}_n$ (resp. $\mathcal{D}/\mathrm{Fil}^1 S_K \mathcal{D}$), we obtain an isomorphism $D_{K_n} := K_n \otimes_{W(k)[1/p]} D \overset{\sim}{\to} \mathcal{D}_n/\mathrm{Fil}^1 S_{K_n}\mathcal{D}_n$ (resp. $D_K := K \otimes_{W(k)[1/p]} D \overset{\sim}{\to} \mathcal{D}/\mathrm{Fil}^1 S_K \mathcal{D}$). By this isomorphism, we identify D_{K_n} (resp. D_K) with $\mathcal{D}_n/\mathrm{Fil}^1 S_{K_n}\mathcal{D}_n$ (resp. $\mathcal{D}/\mathrm{Fil}^1 S_K \mathcal{D}$). Then the filtration $(\mathrm{Fil}^i D_{K_n})_{i \in \mathbb{Z}}$ on D over K_n is given by $\mathrm{Fil}^i D_{K_n} = f_{\pi_n}(\mathrm{Fil}^i \mathcal{D}_n)$. We note that the filtered (φ, N)-module D over K_n defined above is weakly admissible since V is semi-stable (see [CF, §3.4] for the definition of weakly admissibility).

Let $\mathcal{O}_K$ and $\mathcal{O}_{K_n}$ be rings of integers of K and K_n, respectively. We note that there exists a canonical isomorphism $K_n \otimes_{\mathcal{O}_{K_n}} f_{\pi_n}(\mathrm{Fil}^i \mathcal{M}_n) \simeq K_n \otimes_{\mathcal{O}_{K_n}} f_{\pi_n}(\mathrm{Fil}^i \mathcal{N})$ since we have $p^c \mathrm{Fil}^i \mathcal{M}_n \subset \mathrm{Fil}^i \mathcal{N} \subset \mathrm{Fil}^i \mathcal{M}_n$ as submodules of $\mathcal{D}_n$, where $c \geq 0$ is an integer such that $\mathfrak{M}_n/\mathfrak{N}$ is killed by p^c. On the other hand, the canonical isomorphism $S_{K_n}[1/p] \otimes_{S_K[1/p]} \mathcal{D} \simeq \mathcal{D}_n$ induces an isomorphism $S_{K_n} \otimes_{S_K} \mathrm{Fil}^i \mathcal{M} \simeq \mathrm{Fil}^i \mathcal{M}_n$ (cf. Lemma 3.19 (2)), and it gives an isomorphism $\mathcal{O}_{K_n} \otimes_{\mathcal{O}_K} f_\pi(\mathrm{Fil}^i \mathcal{M}) \simeq f_{\pi_n}(\mathrm{Fil}^i \mathcal{M}_n)$. Furthermore, it follows from [Li2, Corollary 3.2.3] that a natural isomorphism $\mathcal{N}[1/p] \simeq \mathcal{D}_n$ preserves filtrations, where $\mathrm{Fil}^i(\mathcal{N}[1/p]) := (\mathrm{Fil}^i \mathcal{N})[1/p]$. This induces $K_n \otimes_{\mathcal{O}_{K_n}} f_{\pi_n}(\mathrm{Fil}^i \mathcal{N}) \simeq \mathrm{Fil}^i D_{K_n}$. (Here, we remark that the argument of §3.2 of *loc. cit.* proceeds even for $p = 2$.) Therefore, if we define a decreasing filtration $(\mathrm{Fil}^i D_K)_{i \in \mathbb{Z}}$ on D_K by $\mathrm{Fil}^i D_K := K \otimes_{\mathcal{O}_K} f_\pi(\mathrm{Fil}^i \mathcal{M})$, then we have a canonical isomorphism

$$K_n \otimes_K \mathrm{Fil}^i D_K \simeq \mathrm{Fil}^i D_{K_n}. \tag{3.3}$$

Note that we know $D_K = \mathrm{Fil}^0 D_K \supset \mathrm{Fil}^1 D_K \supset \cdots \supset \mathrm{Fil}^{r+1} D_K = 0$. Now we recall that D is weakly admissible as a filtered (φ, N)-module over K_n. It follows from (3.3) that D is also weakly admissible as a filtered (φ, N)-module over K, and hence the action of G_{K_n} on V extends to G_K so that it is semi-stable over K. Therefore, we showed that the restriction functor $\mathbf{R}_1 \to \mathbf{R}_2$ is essentially surjective and this finishes a proof of Proposition 3.15.

3.4 $\mathcal{C}^r_{m_0} = \mathcal{C}^r$

Now we are ready to complete a proof of Theorem 2.6. We put $K_{p^\infty} = \bigcup_{i \geq 0} K(\zeta_{p^i})$ and $G_{p^\infty} = \mathrm{Gal}(K_\infty K_{p^\infty}/K_{p^\infty}) \subset \hat{G}_K$. We fix a topological generator τ of G_{p^∞}. We start with the following lemma.

Lemma 3.20. *(1) The field $K_{p^\infty} \cap K_\infty$ coincides with K or K_1.*
(2) If $(p, m_0) \neq (2, 1)$, then $K_{p^\infty} \cap K_\infty = K$.
(3) If $m \geq 2$, then $K_{p^\infty} \cap K_\infty = K$.

Proof. The assertions (1) and (2) are consequences of [Li2, Lemma 5.1.2] and [Li3, Proposition 4.1.5], and so it is enough to show (3). We may assume $p = 2$. Assume that $K_{p^\infty} \cap K_\infty \neq K$. Then we have $K_{p^\infty} \cap K_\infty = K_1$ by (1). Since K_1 is contained in K_{p^∞}, we have $K_1 \subset K(\zeta_{2^\ell})$ for $\ell > m$ large enough. Since $m \geq 2$, the extension $K(\zeta_{2^\ell})/K(\zeta_{2^m})$ is cyclic and thus there exists only one quadratic subextension in it. By definition of m, the extension $K(\zeta_{2^{m+1}})/K(\zeta_{2^m})$ is degree 2. Since the extension K_1/K is totally ramified but $K(\zeta_{2^m})/K$ is unramified, we see that the extension $K_1(\zeta_{2^m})/K(\zeta_{2^m})$ is also degree 2. Therefore, we have $K_1(\zeta_{2^m}) = K(\zeta_{2^{m+1}})$, and then we have $\pi_1 = x\zeta_{2^{m+1}} + y$ with $x, y \in K(\zeta_{2^m})$. Let σ be a non-trivial element in $\mathrm{Gal}(K(\zeta_{2^{m+1}})/K(\zeta_{2^m}))$. We have $-\pi_1 = \sigma(\pi_1) = x\sigma(\zeta_{2^{m+1}}) + y = -x\zeta_{2^{m+1}} + y$. Hence $\pi_1 = x\zeta_{2^{m+1}}$ and we have $v(\pi_1) = v(x)$. Here, v is a valuation of $K(\zeta_{2^{m+1}})$ normalized by $v(K^\times) = \mathbb{Z}$, and we see $v(\pi_1) = 1/2$. Since the extension $K(\zeta_{2^m})/K$ is unramified, we have $v(x) \in \mathbb{Z}$ but this is a contradiction. □

If $(p, m_0) = (2, 1)$ and $m = 1$, we have $m_0 = m$ and then Theorem 2.6 follows immediately from Lemma 3.1. Hence we may assume $(p, m_0) \neq (2, 1)$ or $m \geq 2$. Under this assumption, the above lemma implies $K_{p^\infty} \cap K_\infty = K$. In particular, we have $\hat{G} = G_{p^\infty} \rtimes H_K$ with the relation $g\sigma = \sigma^{\chi(g)} g$ for $g \in H_K$ and $\sigma \in G_{p^\infty}$. Here, χ is the p-adic cyclotomic character. Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G}_K)$ be an object of ${}_{\mathrm{w}}\mathrm{Mod}^{r, \hat{G}_K}_{/\mathfrak{S}_K}$ and put $T = \hat{T}_K(\hat{\mathfrak{M}})$. Our goal is to show that T is an object of $\mathcal{C}^r_{m_0}$. We put $\mathcal{D} = S_K[1/p] \otimes_{\varphi, \mathfrak{S}_K} \mathfrak{M}$ and $D = \mathcal{D}/I_+ S_K[1/p]\mathcal{D}$. Let $s\colon D \hookrightarrow \mathcal{D}$ be a φ-equivariant $W(k)[1/p]$-linear section of the projection $\mathcal{D} \twoheadrightarrow D$ as before, and take a basis $e_1, \ldots, e_d$ of $s(D)$. In $\mathcal{R}_K \otimes_{W(k)[1/p]} s(D) = \mathcal{R}_K \otimes_{\varphi, \mathfrak{S}_K} \mathfrak{M}$, the τ-action with respected to the basis $e_1, \ldots, e_d$ is given by $\tau(e_1, \ldots e_d) = (e_1, \ldots, e_d)A(t)$ for some matrix $A(t) \in GL_d(W(k)[1/p][\![t]\!])$. Moreover, we have $\hat{G}_K(s(D)) \subset (\mathcal{R}_K \cap W(k)[1/p][\![t]\!]) \otimes_{W(k)[1/p]} s(D)$ by [Li1, Lemma 7.1.3]. Here are two remarks. The first one is that, the a-th power $A(t)^a$, a matrix with coefficients in $W(k)[1/p][\![t]\!]$, of $A(t)$ is well-defined for any $a \in \mathbb{Z}_p$. This is because the Galois group $G_{p^\infty} = \tau^{\mathbb{Z}_p} \subset \hat{G}_K$ acts continuously on $\mathcal{R}_K \otimes_{W(k)[1/p]} s(D)$. The second one is that, for any $g \in H_K$, we have $A(\chi(g)t) = A(t)^{\chi(g)}$ by the relation $g\tau = \tau^{\chi(g)} g$. In particular, we have

$$A(0)^{\chi(g)-1} = I_d. \tag{3.4}$$

Here, I_d is the identity matrix. With these notation, it follows from the second paragraph of the proof of [Li3, Theorem 4.2.2] that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over K_ℓ if $A(0)^{p^\ell} = I_d$.

Lemma 3.21. *Let the notation be as above. Then we have $A(0)^{p^{m_0}} = I_d$.*

Proof. First we consider the case where p is odd. Since H_K is canonically isomorphic to $\mathrm{Gal}(K_{p^\infty}/K)$, the image of the restriction to H_K of the p-adic cyclotomic character $\chi\colon \hat{G}_K \to \mathbb{Z}_p^\times$ is equal to

$$\chi(\hat{G}_K) = C \times (1 + p^n \mathbb{Z}_p)$$

where n is a positive integer and $C \simeq \mathrm{Gal}(K(\zeta_p)/K)$ is a finite cyclic group of order prime-to-p.
The case where $m_0 \geq 1$: In this case, it is an easy exercise to check the equality $n = m_0$ and hence we can choose $g \in H_K$ such that $\chi(g) = 1 + p^{m_0}$. Thus the result follows by (3.4).
The case where $m_0 = 0$: In this case, C is non-trivial and hence there exists an element $g \in H_K$ such that $x := \chi(g) - 1$ is a unit of $\mathbb{Z}_p$. By (3.4), we have $A(0)^x = I_d$, and then we obtain $A(0) = I_d$.

Next we consider the case where $p = 2$.
The case where $m_0 \geq 2$: This case is clear since we have $\chi(H_K) = \chi(\hat{G}_K) = 1 + 2^{m_0}\mathbb{Z}_2$.
The case where $m_0 = 1$: In this case, $\chi \bmod 4$ is not trivial. Hence there exists $g \in H_K$ such that $\chi(g) = 3 + 4x$ for some $x \in \mathbb{Z}_2$. By (3.4), we have $A(0)^{2+4x} = I_d$. Since $1 + 2x$ is a unit of $\mathbb{Z}_2$, this gives the desired equation $A(0)^2 = I_d$. □

By the above lemma, we obtain the fact that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable over K_{m_0}. On the other hand, we have already shown that $\mathcal{C}^r$ is a subcategory of $\mathcal{C}^r_m$. Thus there exists a semi-stable $\mathbb{Q}_p$-representation V of G_K whose restriction to G_{K_m} is isomorphic to $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Moreover, Proposition 2.8 implies that V and $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ are isomorphic as representations of $G_{K_{m_0}}$ since they are semi-stable over K_{m_0}. Therefore, we conclude that T is an object of the category $\mathcal{C}^r_{m_0}$. This is the end of a proof of Theorem 2.6.

3.5 Conclusions and more

3.5.1

We summarize our results here. For any finite extension L/K, we denote by $\mathrm{Rep}^{r,L\text{-st}}_{\mathbb{Z}_p}(G_K)$ the category of free $\mathbb{Z}_p$-representations T of G_K which is semi-stable over L with Hodge-Tate weights in $[0, r]$. We define $\mathcal{C}^r_n$ to be the category of free $\mathbb{Z}_p$-representations T of G_K which satisfies the following property: there exists a semi-stable $\mathbb{Q}_p$-representation V of G_K with Hodge-Tate weights in $[0, r]$ such that $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to V as representations of G_{K_n}. By definition $\mathcal{C}^r_n$ is a full subcategory of $\mathrm{Rep}^{r,K_n\text{-st}}_{\mathbb{Z}_p}(G_K)$. Put $m_0 = \max\{i \geq 0 \mid \zeta_{p^i} \in K\}$ and $m = \max\{i \geq 0 \mid \zeta_{p^i} \in K^{\mathrm{ur}}\}$. We have $\mathrm{Rep}^{r,K_m\text{-st}}_{\mathbb{Z}_p}(G_K) = \bigcup_{n\geq 0} \mathrm{Rep}^{r,K_n\text{-st}}_{\mathbb{Z}_p}(G_K)$, $\mathcal{C}^r_m = \bigcup_{n\geq 0} \mathcal{C}^r_n$ (see Remark 2.5). Results of [Li3] and this note give the following diagram (here, "$\subset$" implies an inclusion):

$$\begin{array}{ccccc}
 & & \mathcal{C}^r_m & \xrightarrow{\subset} & \mathrm{Rep}^{r,K_m\text{-st}}_{\mathbb{Z}_p}(G_K) \\
 & & \cup \uparrow & & \cup \uparrow \\
{}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}_K}_{/\mathfrak{S}_K} & \xrightarrow[\hat{T}_K]{\sim} & \mathcal{C}^r_{m_0} & \xrightarrow{\subset} & \mathrm{Rep}^{r,K_{m_0}\text{-st}}_{\mathbb{Z}_p}(G_K) \\
\cup \uparrow & & \cup \uparrow & & \cup \uparrow \\
\mathrm{Mod}^{r,\hat{G}_K}_{/\mathfrak{S}_K} & \xrightarrow[\hat{T}_K]{\sim} & \mathcal{C}^r_0 & = & \mathrm{Rep}^{r,K\text{-st}}_{\mathbb{Z}_p}(G_K)
\end{array}$$

3.5.2

We give a few remarks for the above diagram. Clearly, all the categories in the middle and right vertical lines are same if $m = 0$. On the other hand, if $m \geq 1$, inclusion relations between them are described as follows:

Proposition 3.22. *Suppose $m \geq 1$.*
(1) Suppose $1 \leq n \leq m$. Then the category $\mathcal{C}^r_n$ is strictly larger than $\mathcal{C}^r_{n-1}$. In particular, the category $\mathrm{Rep}^{r,K_n\text{-st}}_{\mathbb{Z}_p}(G_K)$ is strictly larger than $\mathrm{Rep}^{r,K_{n-1}\text{-st}}_{\mathbb{Z}_p}(G_K)$.
(2) Suppose $n, r \geq 1$. Then the category $\mathrm{Rep}^{r,K_n\text{-st}}_{\mathbb{Z}_p}(G_K)$ is strictly larger than $\mathcal{C}^r_n$.
(3) Suppose $n \geq 0$. Then we have $\mathcal{C}^0_n = \mathrm{Rep}^{0,K_n\text{-st}}_{\mathbb{Z}_p}(G_K)$.

Proof. (1) Let T be the induced representation of the rank one trivial $\mathbb{Z}_p$-representation of $G_{K_n(\zeta_{p^n})}$ to G_K, which is an Artin representation. The splitting field of T is $K_n(\zeta_{p^n})$. Since $n \leq m$, the extension $K_n(\zeta_{p^n})/K_n$ is unramified. Thus T is crystalline over K_n. On the other hand, T is not crystalline over K_{n-1} since the extension $K_n(\zeta_{p^n})/K_{n-1}$, the splitting field of $T|_{K_{n-1}}$, is not unramified. (This finishes a proof of the latter assertion.) Let $\rho_T \colon G_K \to GL_{\mathbb{Z}_p}(T) \simeq GL_d(\mathbb{Z}_p)$ be the continuous homomorphism associated with T, where d is the $\mathbb{Z}_p$-rank of T. By the assumption $n \leq m$, we know that $K(\zeta_{p^n}) \cap K_n = K$ and thus we can define a continuous homomorphism $\rho_{T'} \colon G_K \to GL_d(\mathbb{Z}_p)$ by the composite $G_K \twoheadrightarrow \mathrm{Gal}(K(\zeta_{p^n})/K) \simeq \mathrm{Gal}(K_n(\zeta_{p^n})/K_n) \overset{\rho_T}{\hookrightarrow} GL_d(\mathbb{Z}_p)$. Let T' be the free $\mathbb{Z}_p$-module of rank d equipped with a G_K-action by $\rho_{T'}$. Then T' is isomorphic to T as representations of G_{K_n} and furthermore it is crystalline over K. It follows that T is an object of $\mathcal{C}^r_n$.
(2) Since $m \geq 1$, we know that $L := K(\zeta_p)$ is an unramified extension of K. Thus π_K is a uniformizer of L. Consider notations $\mathfrak{S}_L, S_L, \dots$ (resp. $\mathfrak{S}_{L_1}, S_{L_1}, \dots$) with respect to the uniformizer π_K (resp. $\pi_{K,1}$) of L (resp. L_1) and the system $(\pi_{K,n})_{n\geq 0}$ (resp. $(\pi_{K,n+1})_{n\geq 0}$). Let $\mathfrak{M}$ be the rank-2 free Kisin module over $\mathfrak{S}_{L_1}$ of height 1 given by $\varphi(e_1, e_2) = (e_1, e_2)\begin{pmatrix} 1 & u_{L_1} \\ 0 & E_{L_1}(u_{L_1}) \end{pmatrix}$, where $\{e_1, e_2\}$ is a basis of $\mathfrak{M}$. Since $\mathfrak{M}$ is of height 1, there exists a G_{L_1}-stable $\mathbb{Z}_p$-lattice T in a crystalline $\mathbb{Q}_p$-representation of G_{L_1}, coming from a p-divisible group over the integer ring of L_1. We see that $\tilde{T} := \mathrm{Ind}_{G_{L_1}}^{G_K} T$ is crystalline over L_1. Since L_1 is unramified over K_1, $\tilde{T}$ is in fact crystalline over K_1. Furthermore, $\tilde{T}$ does not come from Kisin modules over $\mathfrak{S}_K$ (that is, $\tilde{T}|_{G_{K_\infty}}$ is not isomorphic to $T_{\mathfrak{S}_K}(\mathfrak{N})$ for any Kisin module $\mathfrak{N}$ over $\mathfrak{S}_K$). To check this, it suffices to show that $\tilde{T}$ does not come from Kisin modules over $\mathfrak{S}_L$. Essentially, this has been already shown in [Li3, Example 4.2.3]. Therefore, Corollary 3.16 implies that $\tilde{T}$ is not an object of $\mathcal{C}^r_n$.
(3) We may suppose $n \leq m$. Take any object T of $\mathrm{Rep}^{0,K_n\text{-st}}_{\mathbb{Z}_p}(G_K)$. Since T has only one Hodge-Tate weight zero, the condition $T|_{G_{K_n}}$ is semi-stable implies that $T|_{G_{K_n}}$ is unramified. Thus if we denote by K_T the splitting field of T, then $K_T K_n$ is unramified over K_n.

First we consider the case where K_T contains ζ_{p^n}. In this case, we follow the idea given in the proof of (1). Denote by K' the maximum unramified subextension of $K_T K_n$ over K. Since K_T contains ζ_{p^n}, $K_T K_n/K$ is a Galois extension and hence K'/K is also Galois. Furthermore, it is not difficult to check that the equality $K_T K_n = K' K_n$ holds. Let $\rho_T \colon G_K \to GL_{\mathbb{Z}_p}(T) \simeq GL_d(\mathbb{Z}_p)$ be the continuous homomorphism associated with T, where d is the $\mathbb{Z}_p$-rank of T, and define a continuous homomorphism $\rho_{T'} \colon G_K \to GL_d(\mathbb{Z}_p)$ by the composite $G_K \twoheadrightarrow \mathrm{Gal}(K'/K) \simeq \mathrm{Gal}(K_T K_n/K_n) \overset{\rho_T}{\hookrightarrow} GL_d(\mathbb{Z}_p)$. Let T' be the free $\mathbb{Z}_p$-module of rank d equipped with a G_K-action by $\rho_{T'}$. Then T' is isomorphic to T as representations of G_{K_n} and furthermore, T' is crystalline over K. It follows that T is an object of $\mathcal{C}^0_n$.

Next we consider the general case. Denote by T_0 the induced representation of the rank one trivial $\mathbb{Z}_p$-representation of $G_{K(\zeta_{p^n})}$ to G_K. We define a free $\mathbb{Z}_p$-representation $\tilde{T}$ of G_K by $\tilde{T} := T \oplus T_0$. The splitting fields of $\tilde{T}$ and T_0 are equal to $K_{\tilde{T}} := K_T(\zeta_{p^n})$ and $K(\zeta_{p^n})$, respectively. The representations $\tilde{T}$ and T_0 are objects of $\mathrm{Rep}^{0,K_n\text{-st}}_{\mathbb{Z}_p}(G_K)$. Moreover, the above argument implies that $\tilde{T}$ and T_0 are contained in $\mathcal{C}^0_n$. Therefore, there exist objects $\tilde{\mathfrak{M}}$ and $\mathfrak{M}_0$ of $\mathrm{Mod}^r_{/\mathfrak{S}_K}$ such that $T_{\mathfrak{S}_K}(\tilde{\mathfrak{M}}) = \tilde{T}|_{G_{K_\infty}}$ and $T_{\mathfrak{S}_K}(\mathfrak{M}_0) = T_0|_{G_{K_\infty}}$. Now we recall that the functor $T_{\mathfrak{S}_K}$ is fully faithful. If we denote by $\mathfrak{f} \colon \mathfrak{M}_0 \to \tilde{\mathfrak{M}}$ a (unique) morphism of φ-modules over $\mathfrak{S}_K$ corresponding to the natural projection $\tilde{T} \twoheadrightarrow T_0$, then we obtain a split exact sequence $0 \to \mathfrak{M}_0 \overset{\mathfrak{f}}{\to} \tilde{\mathfrak{M}} \to \mathfrak{M} \to 0$ of φ-modules over $\mathfrak{S}_K$. Here, $\mathfrak{M}$ is the cokernel of $\mathfrak{f}$, which is a finitely generated $\mathfrak{S}_K$-module. Since $\mathfrak{M}$ is a direct summand of $\tilde{\mathfrak{M}}$, it is a projective $\mathfrak{S}_K$-module. This implies that $\mathfrak{M}$ is a free $\mathfrak{S}_K$-module. (Note that, for a finitely generated $\mathfrak{S}_K$-module, it is projective over $\mathfrak{S}_K$ if and only if it is free $\mathfrak{S}_K$ by Nakayama's lemma.) Furthermore, $\mathfrak{M}$ is of height 0 and hence it is an object of $\mathrm{Mod}^0_{/\mathfrak{S}_K}$. Since the functor $T_{\mathfrak{S}_K}$ is exact, we obtain $T_{\mathfrak{S}_K}(\mathfrak{M}) = \ker(T_{\mathfrak{S}_K}(\tilde{\mathfrak{M}}) \overset{T_{\mathfrak{S}_K}(\mathfrak{f})}{\longrightarrow} T_{\mathfrak{S}_K}(\mathfrak{M}_0)) = \ker(\tilde{T} \twoheadrightarrow T_0) = T$. Therefore, T is an object of $\mathcal{C}^0_n$ by Corollary 3.16. □

3.5.3

Assume that $m \geq 1$. Let $n \geq 1$ be an integer and T an object of the category $\mathcal{C}_n^r$. By definition of $\mathcal{C}_n^r$, we have a (unique) semi-stable $\mathbb{Q}_p$-representation V_T of G_K with the property that it is isomorphic to $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as representations of G_{K_n}. It is not clear whether T is stable under the G_K-action of V_T for any T or not. Such a stability problem of Galois actions may sometimes cause obstructions in integral theory, and so the following question should be naturally considered.

Question 3.23. Let the notation be as above. Does the G_K-action of V_T preserves T for any T?

We end this paper by showing an answer to this question.

Proposition 3.24. *(1) If $r = 0$, then Question 3.23 has an affirmative answer.*
(2) If $r \geq 1$, then Question 3.23 has a negative answer.
(3) Let the notation be as above. Suppose $e(r-1) < p-1$ where e is the absolute ramification index of K. If T is potentially crystalline, then the G_K-action of V_T preserves T. Moreover, any G_{K_∞}-stable $\mathbb{Z}_p$-lattice of V_T is stable under the G_K-action.

Proof. (1) (This is a special case of (3).) The result easily follows from the fact that T as in the question is unramified in this case, and that G_{K_n} and the inertia subgroup of G_K generate G_K.
(2) Our goal is to construct an example which gives a negative answer to the question. First we consider the case where $1 \leq n \leq m_0$. Let E_π be the Tate curve over K associated to π. Choose a basis $\{\mathbf{e},\ \mathbf{f}\}$ of the p-adic Tate module $V = V_p(E_\pi)$ of E_π such that the G_K-action on V with respective to this basis is given by

$$g \mapsto \begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix}.$$

Here, $\chi\colon G_K \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character and $c\colon G_K \to \mathbb{Z}_p$ is a map defined by $g(\pi_{K,\ell}) = \zeta_{p^\ell}^{c(g)} \pi_{K,\ell}$ for any $g \in G_K$ and $\ell \geq 1$. Let T_0 be the free $\mathbb{Z}_p$-submodule of V generated by $p^n\mathbf{e}$ and $\mathbf{f}$. This is G_{K_n}-stable but not G_K-stable in V. Now we put $T = \mathrm{Ind}_{G_{K_n}}^{G_K} T_0$ and choose a set $S \subset G_K$ of representatives of the quotient G_K/G_{K_n}. Since K_n/K is Galois, $T|_{G_{K_n}}$ is of the form $\oplus_{\sigma \in S} T_{0,\sigma}$. Here, $T_{0,\sigma}$ is just T_0 as a $\mathbb{Z}_p$-module and is equipped with a σ-twisted G_{K_n}-action, that is, $g.x := (\sigma^{-1} g \sigma)(x)$ for $g \in G_{K_n}$ and $x \in T_{0,\sigma}$. We define elements $\mathbf{e}_\sigma$ and $\mathbf{f}_\sigma$ of $T_{0,\sigma}$ by $\mathbf{e}_\sigma := p^n\mathbf{e}$ and $\mathbf{f}_\sigma := \mathbf{f}$. We define $V_{0,\sigma} := T_{0,\sigma} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and extend the G_{K_n}-action on $V_{0,\sigma}$ to G_K by

$$g(\mathbf{e}_\sigma, \mathbf{f}_\sigma) = (\mathbf{e}_\sigma, \mathbf{f}_\sigma) \begin{pmatrix} \chi(g) & c(\sigma^{-1} g \sigma)/p^n \\ 0 & 1 \end{pmatrix}$$

for $g \in G_K$. By definition the G_K-action on $V_{0,\sigma}$ does not preserve $T_{0,\sigma}$. It is not difficult to check that $V_{0,\sigma}$ is a semi-stable $\mathbb{Q}_p$-representation of G_K with Hodge-Tate weights $\{0,1\}$. If we put $V_T = \oplus_{\sigma \in S} V_{0,\sigma}$, then we have the followings:

- V_T is semi-stable over K with Hodge-Tate weights $\{0,1\}$,
- the natural isomorphism $V_T \simeq T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is compatible with G_{K_n}-actions, and
- the G_K-action on V_T does not preserve T.

This gives a negative answer to Question 1.1 in the case $1 \leq n \leq m_0$.

Next we consider a general case. We may suppose $n = m$. Put $K' = K(\zeta_{p^m})$ and $K'_m = K_m K'$. Then K' is an unramified Galois extension of K and $\max\{i \geq 0 \mid \zeta_{p^i} \in K'\} = m$. Thus the above argument shows that there exists a free $\mathbb{Z}_p$-representation T' of $G_{K'}$ and a semi-stable $\mathbb{Q}_p$-representation $V_{T'}$ of $G_{K'}$ with Hodge-Tate weights $\{0,1\}$ which satisfies the followings:

- there exists an isomorphism $V_{T'} \simeq T' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of $G_{K'_m}$-representations, and
- the $G_{K'}$-action on $V_{T'}$ does not preserve T'.

We regard T' as a $\mathbb{Z}_p$-lattice of $V_{T'}$. We define $T := \mathrm{Ind}_{G_{K'}}^{G_K} T'$ and $V_T := \mathrm{Ind}_{G_{K'}}^{G_K} V_{T'}$. Note that T is naturally regarded as a $\mathbb{Z}_p$-lattice of V. By definition, the $G_{K'}$-action on V_T does not preserve T. In particular, the same holds also for the G_K-action. Since K'/K is unramified, we see that V_T is semi-stable over K. Furthermore, by Mackey's formula, we have natural isomorphisms $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathrm{Ind}_{G_{K'_m}}^{G_{K_m}} (T' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \simeq \mathrm{Ind}_{G_{K'_m}}^{G_{K_m}} V_{T'} \simeq V_T$ of representations of G_{K_m}. Therefore, we conclude that Question 3.23 has a negative answer for any $n \geq 1$.
(3) This is a special case of [Oz, Corollary 4.20]. □

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