Non-existence of certain Galois representations with a uniform tame inertia weight

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Abstract

In this paper, we prove the non-existence of certain semistable Galois representations of a number field. Our consequence can be applied to some geometric problems. For example, we prove a special case of a Conjecture of Rasmussen and Tamagawa, related with the finiteness of the set of isomorphism classes of abelian varieties with constrained prime power torsion.

0 Introduction

Let ℓ be a prime number and K a number field. In this paper, we show the non-existence of certain semistable ℓ-adic Galois representations of the absolute Galois group G_K of K by using remarkable results on the tame inertia weights due to Caruso. Fix non-negative integers n, r and w, and a prime number $\ell_0 \neq \ell$. Put $\bullet := (n, \ell_0, r, w)$. We consider the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ of isomorphism classes of ℓ-adic representations of G_K (Definition 2.4 (2)). This set is related with the dual of $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$, where X is a proper smooth scheme over K which has everywhere semistable reduction and has good reduction at a place of K above ℓ_0. Our main result in this paper is

Theorem 0.1 (= Theorem 2.11)**.** *Suppose that w is odd or $w > 2r$. Then there exists an explicit constant C depending only on K, n, ℓ_0, r and w such that $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ is empty for any prime number $\ell > C$ which does not split in K.*

Theorem 0.1 comes from a relation between the tame inertia weights and eigenvalues of Frobenius action (Proposition 2.8). As a by-product of the above theorem, we obtain some approaches to algebraic geometry. For example, our result gives an application to a special case of the Rasmussen-Tamagawa conjecture ([RT]) related with the finiteness of the set of isomorphism classes of abelian varieties with constrained prime power torsion.

Now we describe an organization of this paper. In Section 1, we recall some results on integral p-adic Hodge theory given by Caruso [Ca]. In Section 2, we give explicit values of the tame inertia weights for certain semistable Galois representations and prove our non-existence theorem. In Section 3, we apply our consequence for some geometric problems.

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Notation:
For a prime number ℓ and a topological group G, an *ℓ-adic representation of G* (resp. $\mathbb{F}_\ell$-representation) is a finite-dimensional $\mathbb{Q}_\ell$-vector space V (resp. $\mathbb{F}_\ell$-vector space V) equipped with a continuous and linear G-action. For such a representation V, we denote by $V^\vee$ the dual of V, that is,

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$V^\vee := \mathrm{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell)$ (resp. $V^\vee := \mathrm{Hom}_{\mathbb{F}_\ell}(V, \mathbb{F}_\ell)$) with the G-action defined by $g.f(v) := f(g^{-1}.v)$ for $f \in V^\vee, g \in G$ and $v \in V$. For any scheme X over a commutative ring R and an R-algebra R', we denote the fiber product $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$ by $X_{R'}$.

1 Tame inertia weights of semistable representations

In this section, we recall the definition of the tame inertia weights (cf. [Se], Section 1) and Caruso's work for the tame inertia weights of a residual representation of semistable Galois representations (cf. [Ca]). Let K_λ be a complete discrete valuation field of characteristic zero with perfect residue field k of positive characteristic ℓ and G_{K_λ} its absolute Galois groups. Let e be the absolute ramification index of K_λ. The tame inertia weights of an ℓ-adic semistable Galois representation of G_{K_λ} with Hodge-Tate weights in $[0, r]$ have remarkable properties if $er < \ell - 1$. For example, Serre conjectured in [Se] that the tame inertia weights on the Jordan-Hölder quotients of a residual representation of the r-th ℓ-adic étale cohomology group $H^r_{\text{ét}}(X_{\bar{K}_\lambda}, \mathbb{Q}_\ell)$ of a proper smooth scheme X over K_λ are between 0 and er. Caruso proved this Serre's conjecture in [Ca] by using the integral p-adic Hodge theory. As an another example, in [CS], Caruso and Savitt proposed the tame inertia polygon of an ℓ-adic semistable Galois representation of G_{K_λ}, and they showed that this polygon has good relations with the Hodge polygon and the Newton polygon introduced in [Fo].

1.1 Tame inertia weights

We denote by I_λ the inertia subgroup of G_{K_λ}, I_w its wild inertia subgroup and $I_t := I_\lambda / I_w$ the tame inertia group. Let V be an h-dimensional irreducible $\mathbb{F}_\ell$-representation of I_λ and fix a separable closure $\bar{\mathbb{F}}_\ell$ of $\mathbb{F}_\ell$. By the irreducibility, the action of I_λ on V factors through I_t and thus we can regard V as a representation of I_t. Applying Schur's lemma, we see that $\mathbb{E} := \mathrm{End}_{I_t}(V)$ is the finite field of order ℓ^h. Moreover, the representation V inherits a structure of a 1-dimensional $\mathbb{E}$-representation of I_t by the natural manner. This representation is given by a character $\rho\colon I_t \to \mathbb{E}^\times$. Choose any isomorphism $f\colon \mathbb{E} \to \mathbb{F}_{\ell^h}$ and consider the composition $\rho_f\colon I_t \xrightarrow{\rho} \mathbb{E}^\times \xrightarrow{f} \mathbb{F}_{\ell^h}^\times$:

$$\begin{tikzcd} I_t \arrow[r, "\rho"'] \arrow[rr, bend left, "\rho_f"] \arrow[dr] & \mathbb{E}^\times \arrow[r, "f"', "\simeq"] \arrow[d, hook] & \mathbb{F}_{\ell^h}^\times. \\ & GL(V) = \mathrm{End}_{\mathbb{F}_\ell}(V)^\times & \end{tikzcd}$$

Denote by $\mu_{\ell^h-1}(\bar{K}_\lambda)$ the set of (ℓ^h-1)-st roots of unity in a separable closure $\bar{K}_\lambda$ of K_λ. Consider an isomorphism $\mu_{\ell^h-1}(\bar{K}_\lambda) \simeq \mathbb{F}_{\ell^h}^\times$ coming from a surjection $\mathcal{O}_{\bar{K}_\lambda} \to \bar{\mathbb{F}}_\ell$, where $\mathcal{O}_{\bar{K}_\lambda}$ is the integer ring of $\bar{K}_\lambda$, and take the following fundamental character of level h:

$$\begin{aligned} \theta_h\colon I_t &\to \mu_{q-1}(\bar{K}_\lambda) \simeq \mathbb{F}_{\ell^h}^\times. \\ \sigma &\mapsto \frac{\eta^\sigma}{\eta} \end{aligned}$$

Here η is a (ℓ^h-1)-st root of a uniformizer of K_λ. It is easy to check that $\theta_h^{1+\ell+\cdots+\ell^{h-1}} = \theta_1$, $\theta_h^{\ell^h-1} = 1$ and, with respect to h embeddings $\mathbb{F}_{\ell^h} \hookrightarrow \bar{\mathbb{F}}_\ell$, all the fundamental characters are given by $\theta_{h,0}(:= \theta_h), \theta_{h,1}, \theta_{h,2}, \ldots, \theta_{h,h-1}$, where $\theta_{h,i} = \theta_{h,i-1}^\ell$ for $0 \le i \le h-1$ and $\theta_{h,0} = \theta_{h,h-1}^\ell$. It is known that θ_1^e coincides with the mod ℓ cyclotomic character ([Se], Section 1.8, Proposition 8). Since I_t is pro-cyclic and $\mathrm{Im}(\theta_h) = \mathbb{F}_{\ell^h}^\times$, there exists an integer $n_f \in \{0, 1, \ldots, \ell^h - 2\}$ such that $\rho_f = \theta_h^{n_f}$. If we decompose $n_f = n_0 + n_1\ell + n_2\ell^2 + \cdots + n_{h-1}\ell^{h-1}$ with integers $0 \le n_i \le \ell - 1$ for any i, then we can see that the set $\{n_0, n_1, n_2, \ldots, n_{h-1}\}$ is independent of the choice of f.

Definition 1.1. We call these numbers $n_0, n_1, n_2, \ldots, n_{h-1}$ the *tame inertia weights of* V. In general, for any $\mathbb{F}_\ell$-representation V of I_λ, the tame inertia weights of V are the numbers of the tame inertia weights of all the Jordan-Hölder quotients of V.

Example 1.2. Suppose that k is algebraically closed. Let E be an elliptic curve over K_λ with semistable reduction. If E has supersingular reduction, assume $e = 1$. Then the tame inertia weights of $E[\ell]$ are 0 and e (cf. [Se], Section 1, Proposition 11 and 12).

Definition 1.3. Let V be an ℓ-adic representation of G_{K_λ}. The tame inertia weights of V is the tame inertia weights of a residual representation of $V|_{I_\lambda}$.

The above definition is independent of the choice of a residual representation of V by the Brauer-Nesbitt theorem.

Definition 1.4. Let w be an integer with $0 \leq w < \ell - 1$ and V be an n-dimensional ℓ-adic representation of G_{K_λ}. Denote by $w_1 \leq w_2 \leq \cdots \leq w_n$ all the tame inertia weights of V. We say that V is *of uniform tame inertia weight w* if $w_1 = w_2 = \cdots = w_n = w$.

1.2 Caruso's Result

Fix an integer $r \geq 0$ such that $er < \ell - 1$. We use the ring S and the category $\mathrm{Mod}^{r,\Phi,N}_{/S_\infty}$ of finite torsion S-modules equipped with some additional structures as in Section 1 of [Ca] without giving the precise definitions. The category $\mathrm{Mod}^{r,\Phi,N}_{/S_\infty}$ is just the category $\underline{\mathcal{M}}^r$ given in *op.cit.* The category $\mathrm{Mod}^{r,\Phi,N}_{/S_\infty}$ is an abelian category (cf. [Ca], Section 3.5). We denote by $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_\ell}(G_{K_\lambda})^r$ (resp. $\mathrm{Rep}_{\mathbb{Z}_\ell}(G_{K_\lambda})_{\mathrm{tors}}$) the category of G_{K_λ}-stable $\mathbb{Z}_\ell$-lattices of semistable ℓ-adic representations of G_{K_λ} with Hodge-Tate weights in $[0, r]$ (resp. the category of finite torsion $\mathbb{Z}_\ell$-modules with a continuous G_{K_λ}-action). Denote by $\mathrm{Mod}^{r,\Phi,N}_{/S}$ the category of strongly divisible modules over S of weight r (cf. [Ca], Section 7.1). There exist the following two contravariant functors

$$T_{\mathrm{st}} : \mathrm{Mod}^{r,\Phi,N}_{/S} \to \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_\ell}(G_{K_\lambda})^r$$

and

$$T_{\mathrm{st}} : \mathrm{Mod}^{r,\Phi,N}_{/S_\infty} \to \mathrm{Rep}_{\mathbb{Z}_\ell}(G_{K_\lambda})_{\mathrm{tors}}$$

satisfying good properties. For example,

(1) (cf. [Ca], Theorem 1.0.5) The 1st T_{st} is an isomorphism,

(2) (cf. [Ca], Theorem 1.0.4) The 2nd T_{st} is exact and fully faithful, and its essential image is stable under taking sub-objects and quotient objects.

If $\mathcal{M} \in \mathrm{Mod}^{r,\Phi,N}_{/S_\infty}$ is isomorphic to $S/\ell^{n_1}S \oplus S/\ell^{n_2}S \oplus \cdots \oplus S/\ell^{n_d}S$ as S-modules, then $T_{\mathrm{st}}(\mathcal{M})$ is isomorphic to $\mathbb{Z}_\ell/\ell^{n_1}\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell/\ell^{n_2}\mathbb{Z}_\ell \oplus \cdots \oplus \mathbb{Z}_\ell/\ell^{n_d}\mathbb{Z}_\ell$ as $\mathbb{Z}_\ell$-modules ([Ca], Proposition 6.4.5). By the definition of strongly divisible modules, we see that, for any strongly divisible module $\tilde{\mathcal{M}}$ and $n \geq 0$, the quotient $\tilde{\mathcal{M}}/\ell^n\tilde{\mathcal{M}}$ is an object of $\mathrm{Mod}^{r,\Phi,N}_{/S_\infty}$ and the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Mod}^{r,\Phi,N}_{/S} \arrow[r, "T_{\mathrm{st}}"] \arrow[d, "\bmod \ell^n"] & \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_\ell}(G_{K_\lambda})^r \arrow[d, "\bmod \ell^n"] \\
\mathrm{Mod}^{r,\Phi,N}_{/S_\infty} \arrow[r, "T_{\mathrm{st}}"] & \mathrm{Rep}_{\mathbb{Z}_\ell}(G_{K_\lambda})_{\mathrm{tors}}.
\end{tikzcd}$$

If k is algebraically closed and $\mathcal{M} \in \mathrm{Mod}^{r,\Phi,N}_{/S_\infty}$ is a simple object, then $T_{\mathrm{st}}(\mathcal{M})$ is an irreducible $\mathbb{F}_\ell$-representation of G_{K_λ} and its tame inertia weights are between 0 and er ([Ca], Theorem 1.0.3). By using the above facts, we can show the following important theorem:

Theorem 1.5 ([Ca])**.** *Let $T_\ell \in \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_\ell}(G_{K_\lambda})^r$ and $\bar{T}_\ell = T_\ell/\ell T_\ell$ its residual representation. Then the tame inertia weights of $\bar{T}_\ell|_{I_\lambda}$ are between 0 and er.*

Proof. We may assume that k is algebraically closed. Choose the strongly divisible module $\tilde{\mathcal{M}}$ corresponding to T_ℓ via T_{st}. Then $\mathcal{M} := \tilde{\mathcal{M}}/\ell\tilde{\mathcal{M}}$ is contained in $\mathrm{Mod}^{r,\Phi,N}_{/S_\infty}$ and $T_{\mathrm{st}}(\mathcal{M})$ is isomorphic to $\bar{T}_\ell$. We identify $T_{\mathrm{st}}(\mathcal{M})$ with $\bar{T}_\ell$. Since the essential image of $T_{\mathrm{st}}\colon \mathrm{Mod}^{r,\Phi,N}_{/S_\infty} \to \mathrm{Rep}_{\mathbb{Z}_\ell}(G_{K_\lambda})_{\mathrm{tors}}$ is stable under sub-quotient, any Jordan-Hölder quotient of $\bar{T}_\ell$ is isomorphic to the representation of the form $T_{\mathrm{st}}(\mathcal{M}')$ for some $\mathcal{M}' \in \mathrm{Mod}^{r,\Phi,N}_{/S_\infty}$. The object $\mathcal{M}'$ is simple because the functor T_{st} is exact and fully faithful. Therefore, we obtain the desired result. $\square$

Remark 1.6. In fact, we do not need the assumption $er < \ell - 1$ for Theorem 1.5 (the case $er \geq \ell - 1$ is trivial).

2 Non-existence theorems

In this section, we calculate the tame inertia weights of ℓ-adic representations with certain geometric and filtration conditions for a prime number ℓ large enough. As a result, we show the non-existence theorems of certain Galois representations.

Let K be a finite extension over $\mathbb{Q}$ and fix an algebraic closure $\bar{K}$ of K. We put $G_K := \mathrm{Gal}(\bar{K}/K)$, the absolute Galois group of K. Let ℓ be a prime number. For any finite place v of K, we denote by G_v and I_v the decomposition group and its inertia subgroup at v, respectively. Furthermore, we denote by e_v the absolute ramification index at v, q_v the order of the residue field of v and Fr_v the arithmetic Frobenius at v. For a place λ of K above ℓ, we identify G_λ with the absolute Galois group G_{K_λ} of a λ-adic completion K_λ of K via a fixed embedding $\bar{K} \hookrightarrow \bar{K}_\lambda$, where $\bar{K}_\lambda$ is an algebraic closure of K_λ.

Definition 2.1. Let λ be a place of K above ℓ and V an ℓ-adic representation of G_K. The *tame inertia weights of V at λ* is the tame inertia weights of $V|_{G_\lambda}$ (cf. Definition 1.3). For an integer $0 \leq w < \ell - 1$, we say that V is *of uniform tame inertia weight w at λ* if $V|_{G_\lambda}$ is of uniform tame inertia weight w (cf. Definition 1.4).

2.1 Geometric and filtration conditions

We define the set of representations we mainly consider throughout this section. We fix non-negative integers n, r, w and $\bar{w}$, and a prime number ℓ_0 different from ℓ. Let χ_ℓ be the mod ℓ cyclotomic character. Take an n-dimensional ℓ-adic representation V of G_K and denote by $\bar{V}$ its residual representation. Now we consider the following geometric conditions (G-1), (G-2), (G-2)$'$ and (G-3), and filtration conditions (F-1) and (F-2):

(G-1) For any place λ of K above ℓ, the representation $V|_{G_\lambda}$ is semistable and has Hodge-Tate weights in $[0, r]$.

(G-2) For some places λ_0 of K above ℓ_0, the representation V is unramified at λ_0 and the characteristic polynomial $\det(T - \mathrm{Fr}_{\lambda_0}|V)$ has rational integer coefficients. Furthermore, there exists non-negative integers $w_1(V), w_2(V), \ldots, w_n(V)$ such that $w_1(V)+w_2(V)+\cdots+w_n(V) \leq \bar{w}$ and the roots of the above characteristic polynomial have complex absolute values $q_{\lambda_0}^{w_1(V)/2}, q_{\lambda_0}^{w_2(V)/2}, \ldots, q_{\lambda_0}^{w_n(V)/2}$ for every embedding $\bar{\mathbb{Q}}_\ell$ into $\mathbb{C}$.

(G-2)$'$ The condition (G-2) holds and $w_1(V) = w_2(V) = \cdots = w_n(V) = w$.

(G-3) For any finite place λ of K not above ℓ, the action of I_λ on $\bar{V}$ is unipotent.

(F-1) The representation $\bar{V}$ has a filtration of G_K-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_n = \bar{V}$$

such that $\bar{V}_k$ has dimension k for each $1 \leq k \leq n$.

(F-2) The condition (F-1) holds. Moreover, for each $1 \leq k \leq n$, the G_K-action on the quotient $\bar{V}_k/\bar{V}_{k-1}$ is given by $g.\bar{v} = \chi_\ell^{a_k}(g)\bar{v}$ for some $0 \leq a_k \leq \ell - 2$.

If an ℓ-adic representation V satisfies the condition (F-1), then we say that V is *of residually Borel*. We note that it is independent of the choice of a residual representation $\bar{V}$ of V whether the filtration conditions (F-1) and (F-2) hold or not. If $n = 2$, then (F-1) is equivalent to the condition that $\bar{V}$ is reducible.

Example 2.2. Suppose $w \leq r$. Let X be a proper smooth scheme over K which has everywhere semistable reduction and has good reduction at some places of K above ℓ_0. Then the dual $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)^\vee$ of the w-th ℓ-adic étale cohomology group of X satisfies the geometric conditions (G-1), (G-2)$'$ and (G-3).

Proposition 2.3. *Let X be a proper smooth scheme over K and w an odd integer. Denote by S_X the finite set of prime numbers p such that X has bad reduction at some place of K above p. Then, there exists a finite extension L of K such that, for any $\ell \notin S_X$, the ℓ-adic representation $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)$ of G_L is semistable at all finite places.*

In particular, we have the following: Let X and L be as above. Fix a prime number $\ell_0 \notin S_X$ and take a prime number ℓ such that $\ell \neq \ell_0$ and $\ell \notin S_X$. Then $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)^\vee$ satisfies (G-1), (G-2)$'$ and (G-3) as a representation of G_L.

Proof of Proposition 2.3. If we admit the semistable conjecture for X, then we can prove this proposition easily. However, we can obtain the desired result without using the semistable conjecture as below: For any algebraic extension K' of K, denote by $S_{X,K'}$ the set of places of K' which is above one of the prime numbers in S_X. Take any place $v \in S_{X,K}$. By de Jong's alteration theorem ([dJ], Theorem 6.5), there exist a finite extension K'_v of K_v, a proper strictly semistable scheme $\mathcal{Y}^v$ over $\mathcal{O}_{K'_v}$ and a morphism $\mathcal{Y}^v \to \mathcal{X}$ compatible with $\mathrm{Spec}(\mathcal{O}_{K'_v}) \to \mathrm{Spec}(\mathcal{O}_{K_v})$ such that the morphism $f \colon \mathcal{Y}^v \to \mathcal{X}_{\mathcal{O}_{K'_v}}$ induced by the above morphism is an étale alteration (see also [Ts], Theorem A3). Here $\mathcal{X}$ is a proper flat model of X_{K_v} over $\mathcal{O}_{K_v}$. Such a model always exists by the compactification theorem of Nagata. Take any prime number ℓ'. If we denote by f_* and f^* the induced homomorphisms $H^w_{\text{ét}}(\mathcal{Y}^v_{\bar{K}'_v}, \mathbb{Q}_{\ell'}) \to H^w_{\text{ét}}(X_{\bar{K}_v}, \mathbb{Q}_{\ell'})$ and $H^w_{\text{ét}}(X_{\bar{K}_v}, \mathbb{Q}_{\ell'}) \to H^w_{\text{ét}}(\mathcal{Y}^v_{\bar{K}'_v}, \mathbb{Q}_{\ell'})$ respectively, then the map $f_* \circ f^*$ is the multiplication by $\deg(f)$. In particular, the map f^* is injective. Thus we may consider that $H^w_{\text{ét}}(X_{\bar{K}'_v}, \mathbb{Q}_{\ell'})$ is a sub-representation of $H^w_{\text{ét}}(\mathcal{Y}^v_{\bar{K}'_v}, \mathbb{Q}_{\ell'})$. Now take a finite extension $K(v)$ of K and a place $w(v)$ of $K(v)$ above v such that $K(v)_{w(v)} = K'_v$, where $K(v)_{w(v)}$ is the $w(v)$-adic completion of $K(v)$. The existence of $K(v)$ and $w(v)$ is an easy consequence of [La], Chapter II, Section 2, Proposition 4. We denote by L the Galois closure, over K, of the field generated by all $K(v)$. Here v runs through all the places of K in $S_{X,K}$. Now we take a prime number $\ell \notin S_X$. It suffices to show that the ℓ-adic representation $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)$ of G_L is everywhere semistable. Take any finite place w_L of L. If $w_L \notin S_{X,L}$, then X has good reduction at w_L and in particular $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)$ is semistable at w_L. Suppose $w_L \in S_{X,L}$. We denote the restriction of w_L to K by v. Take $\mathcal{Y}^v$ and the place $w(v)$ of $K(v)$ as above. Furthermore, we take a place w'_L of L above $w(v)$. Since the action of $I_{w'_L}$ is unipotent on $H^w_{\text{ét}}(\mathcal{Y}^v_{\bar{L}}, \mathbb{Q}_\ell)$, we have that the action of $I_{w'_L}$ on $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unipotent, too. Since the inertia subgroup $I_{w'_L}$ conjugates with I_{w_L} by the element of G_K, we see that the action of I_{w_L} on $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is also unipotent, that is, $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)$ is semistable at w_L. This finishes the proof. $\square$

Definition 2.4. Put $\circ := (n, \ell_0, r, \bar{w})$ and $\bullet := (n, \ell_0, r, w)$.
(1) We denote by $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ_{\mathrm{cycl}}$ (resp. $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet_{\mathrm{cycl}}$) the set of isomorphism classes of n-dimensional ℓ-adic representations V of G_K which satisfy (G-1), (G-2) and (F-2) (resp. (G-1), (G-2)$'$ and (F-2)).
(2) We denote by $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ$ (resp. $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$) the set of isomorphism classes of n-dimensional ℓ-adic representations V of G_K which satisfy (G-1), (G-2), (G-3) and (F-1) (resp. (G-1), (G-2)$'$, (G-3) and (F-1)).

Clearly, we have

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\circ}_{\mathrm{cycl}} & \subset & \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\circ} \\ \cup & & \cup \\ \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}_{\mathrm{cycl}} & \subset & \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}, \end{array}$$

where $\bullet = (n, \ell_0, r, w)$ and $\circ = (n, \ell_0, r, \bar{w})$ for any $nw \leq \bar{w}$.

Our main concern in this section is the following question:

Question 2.5. *Does there exist a constant C which depends on K and $\bullet$ (or $\circ$) such that the sets defined in Definition 2.4 are empty for $\ell > C$? If the answer is positive, how can we evaluate such a constant C?*

Remark 2.6 (Trivial case). Take a representation $V \in \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$. By (G-2), the complex absolute value of the determinant of Fr_{v_0} acting on V is $q_{v_0}^{nw/2}$ and this must be an integer. From this fact, if n and w are odd and the extension $K/\mathbb{Q}$ is Galois of an odd degree, then $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$ is empty for any prime $\ell \neq \ell_0$. As this example, there exist lots of pairs of $(K, \bullet)$ (resp. $(K, \circ)$) such that $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$ (resp. $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\circ}$) is empty for a prime ℓ (large enough). We hope to know “non-trivial cases” of the emptiness of the sets given in Definition 2.4.

2.2 Main results

We denote by d, d_K and h_K^+ the extension degree of K over $\mathbb{Q}$, the discriminant of K and the narrow class number of K, respectively. Put $M := \max\{nr, \bar{w}/2\}$ and

$$c_n := \begin{cases} \binom{n}{n/2} & \text{if } n \text{ is even,} \\ \binom{n}{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

Clearly this is equal to $\max\{\binom{n}{m} \mid 0 \leq m \leq n\}$. Now we put

$$\begin{aligned} &\varepsilon_1 := dM, \quad \varepsilon_2 := d\varepsilon_1, \quad \varepsilon_1' := dh_K^+ M, \quad \varepsilon_2' := d\varepsilon_1', \\ &C_1 := C_1(d, \bullet) := 2c_n \ell_0^{\varepsilon_1}, \quad C_2 := C_2(d, \bullet) := 2c_n \ell_0^{\varepsilon_2}, \\ &C_1' := C_1'(K, \bullet) := 2c_n \ell_0^{\varepsilon_1'}, \quad C_2' := C_2'(K, \bullet) := 2c_n \ell_0^{\varepsilon_2'}. \end{aligned}$$

The following two propositions play an essential role for our main results.

Proposition 2.7. *Any ℓ-adic representation V in the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\circ}_{\mathrm{cycl}}$ has tame inertia weights $e_\lambda w_1(V)/2, e_\lambda w_2(V)/2, \ldots, e_\lambda w_n(V)/2$ at any place λ of K above ℓ under any one of the following situations:*

(a) $\ell \nmid d_K$ *and* $\ell > C_1$;

(b) $\ell > C_2$.

Proposition 2.8. *Suppose that ℓ is a prime number which does not split in K. Any ℓ-adic representation V in the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\circ}$ has tame inertia weights $e_\lambda w_1(V)/2, e_\lambda w_2(V)/2, \ldots, e_\lambda w_n(V)/2$ at the unique place λ of K above ℓ under any one of the following situations:*

(a) $\ell \nmid d_K$ *and* $\ell > C_1'$;

(b) $\ell > C_2'$.

To prove these propositions, we need the following lemma:

Lemma 2.9. *Let $s, t_1, t_2, \ldots, t_n$ and u be non-negative integers such that $0 \leq s \leq u$ and $0 \leq t_k \leq ru$ for all k. Let V be an n-dimensional ℓ-adic representation of G_K which satisfies* (G-2)*. Decompose $\det(T - \mathrm{Fr}_{\lambda_0}|V) = \prod_{1 \leq k \leq n}(T - \alpha_k)$. If the set $\{\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s\}$ coincides with the set $\{q_{\lambda_0}^{t_1}, q_{\lambda_0}^{t_2}, \ldots, q_{\lambda_0}^{t_n}\}$ in $\bar{\mathbb{F}}_\ell$ and $\ell > 2c_n \ell_0^{dMu}$, then $\{\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s\} = \{q_{\lambda_0}^{t_1}, q_{\lambda_0}^{t_2}, \ldots, q_{\lambda_0}^{t_n}\}$. In particular, we obtain $\{sw_1(V)/2, sw_2(V)/2, \ldots, sw_n(V)/2\} = \{t_1, t_2, \ldots, t_n\}$.*

Proof. We basically follow the proof by the method which has been pointed out by Rasmussen and Tamagawa. Let us denote by $S_m(x_1, x_2, \ldots, x_n)$ the elementary symmetric polynomial of degree m with n-indeterminates $x_1, x_2, \ldots, x_n$ for $0 \le m \le n$, that is,

$$\prod_{1\le k\le n}(T - x_k) = \sum_{0\le m\le n} S_m(x_1, x_2, \ldots, x_n)T^{n-m}.$$

For any $0 \le m \le n$, the condition (G-2) implies that $S_m(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a rational integer for all m and hence $S_m(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s)$, which is a symmetric polynomial of $\alpha_1, \alpha_2, \ldots, \alpha_n$, is also a rational integer. On the other hand, we have

$$\begin{aligned}|S_m(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s)| &\le \sum_{1\le s_1<\cdots<s_m\le n} (q_{\lambda_0}^{(w_{s_1}(V)+\cdots+w_{s_m}(V))/2})^s \\ &\le \sum_{1\le s_1<\cdots<s_m\le n} (q_{\lambda_0}^{\bar{w}/2})^s = \binom{n}{m} (q_{\lambda_0}^{\bar{w}/2})^s \le c_n \ell_0^{dMu}\end{aligned}$$

and

$$\begin{aligned}|S_m(q_{\lambda_0}^{t_1}, q_{\lambda_0}^{t_2}, \ldots, q_{\lambda_0}^{t_n})| &\le \sum_{1\le s_1<\cdots<s_m\le n} q_{\lambda_0}^{t_{s_1}+\cdots+t_{s_m}} \\ &\le \sum_{1\le s_1<\cdots<s_m\le n} q_{\lambda_0}^{nru} = \binom{n}{m} q_{\lambda_0}^{nru} \le c_n \ell_0^{dMu}\end{aligned}$$

by (G-2), where $|\cdot|$ is the complex absolute value. Since we have $S_m(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s) \equiv S_m(q_{\lambda_0}^{t_1}, q_{\lambda_0}^{t_2}, \ldots, q_{\lambda_0}^{t_n})$ mod ℓ and $\ell > 2c_n\ell_0^{dMu}$, we obtain

$$S_m(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s) = S_m(q_{\lambda_0}^{t_1}, q_{\lambda_0}^{t_2}, \ldots, q_{\lambda_0}^{t_n})$$

for all m. This implies

$$\begin{aligned}\prod_{1\le k\le n}(T - \alpha_k^s) &= \sum_{0\le m\le n} S_m(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s)T^{n-m} \\ &= \sum_{0\le m\le n} S_m(q_{\lambda_0}^{t_1}, q_{\lambda_0}^{t_2}, \ldots, q_{\lambda_0}^{t_n})T^{n-m} \\ &= \prod_{1\le k\le n}(T - q_{\lambda_0}^{t_k})\end{aligned}$$

and thus we finish the proof. $\square$

Now we start the proofs of Proposition 2.7 and 2.8. Take any representation V which is an element of the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ$ and denote its residual representation by $\bar{V}$. Then the representation $\bar{V}$ has a filtration of G_K-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_n = \bar{V}$$

such that $\bar{V}_k$ has dimension k for each $1 \le k \le n$. We denote by $\psi_k \colon G_K \to \mathbb{F}_\ell^\times$ the character corresponding to the action of G_K on the quotient $\bar{V}_k/\bar{V}_{k-1}$ for each $1 \le k \le n$. Take any place λ of K above ℓ. By Theorem 1.5, we obtain $\psi_k = \theta_{1,\lambda}^{b_{k,\lambda}}$ on I_λ for some integer $0 \le b_{k,\lambda} \le e_\lambda r$, where $\theta_{1,\lambda} \colon I_\lambda \to \mathbb{F}_\ell^\times$ is the fundamental character of level one at λ. Take a place λ_0 of K above ℓ_0 as in (G-2) and decompose $\det(T - \mathrm{Fr}_{\lambda_0}|V) = \prod_k (T - \alpha_k)$. Then, we see

$$\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \{\psi_1(\mathrm{Fr}_{\lambda_0}), \psi_2(\mathrm{Fr}_{\lambda_0}), \ldots, \psi_n(\mathrm{Fr}_{\lambda_0})\} \qquad (*)$$

in $\bar{\mathbb{F}}_\ell$.

Proof of Proposition 2.7. Assume that V is an element of the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\circ}_{\mathrm{cycl}}$. Then we may suppose $\psi_k = \chi_\ell^{a_k}$ for any k by (F-2). The relation $\chi_\ell^{a_k} = \theta_{1,\lambda}^{b_{k,\lambda}}$ on I_λ implies $\theta_{1,\lambda}^{e_\lambda a_k} = \theta_{1,\lambda}^{b_{k,\lambda}}$ and thus $e_\lambda a_k \equiv b_{k,\lambda} \bmod \ell - 1$. Hence we have $\chi_\ell^{e_\lambda a_k} = \chi_\ell^{b_{k,\lambda}}$ on G_K and thus the set $\{\alpha_1^{e_\lambda}, \alpha_2^{e_\lambda}, \dots, \alpha_n^{e_\lambda}\}$ coincides with the set $\{q_{\lambda_0}^{b_{1,\lambda}}, q_{\lambda_0}^{b_{2,\lambda}}, \dots, q_{\lambda_0}^{b_{n,\lambda}}\}$ in $\bar{\mathbb{F}}_\ell$ by $(*)$. By Lemma 2.9, we have

$$\{e_\lambda w_1(V)/2, \dots, e_\lambda w_n(V)/2\} = \{b_{1,\lambda}, \dots, b_{n,\lambda}\}$$

if $\ell > 2B_n \ell_0^{dMe_\lambda}$. Since $e_\lambda \le d$ and $e_\lambda = 1$ if $\ell \nmid d_K$, we have the desired result. □

Proof of Proposition 2.8. We note that each ψ_k is unramified away from ℓ by (G-3). Now we assume that any one of the following conditions (A) or (B) holds:

(A) $\ell \nmid d_K$;

(B) No additional assumptions.

Setting $b'_k := b_{k,\lambda}/e_\lambda \in \mathbb{Q}$, we have $0 \le b'_k \le r$. We note that, if we put

$$D := \begin{cases} 1 & \text{under (A)}, \\ d & \text{under (B)}, \end{cases}$$

then we see $D/e_\lambda \in \mathbb{Z}$. Since $\psi_k = \theta_{1,\lambda}^{b_{k,\lambda}}$ on I_λ, we see that $\psi_k^{e_\lambda}\chi_\ell^{-b_{k,\lambda}}$ is trivial on I_λ and thus $(\psi_k^{e_\lambda}\chi_\ell^{-b_{k,\lambda}})^{D/e_\lambda} = \psi_k^D \chi_\ell^{-b'_k D}$ is also trivial on I_λ. Since the characters ψ_k and χ_ℓ are unramified away from ℓ, this implies that $\psi_k^D \chi_\ell^{-b'_k D}$ is unramified at all finite places of K (recall that ℓ does not split in K). By class field theory, it follows

$$\psi_k^{Dh_K^+} = \chi_\ell^{b'_k D h_K^+}$$

on G_K. Recall that h_K^+ is the narrow class number of K. Thus we have that the set $\{\alpha_1^{Dh_K^+}, \alpha_2^{Dh_K^+}, \dots, \alpha_n^{Dh_K^+}\}$ coincides with the set $\{q_{\lambda_0}^{b'_1 Dh_K^+}, q_{\lambda_0}^{b'_2 Dh_K^+}, \dots, q_{\lambda_0}^{b'_n Dh_K^+}\}$ in $\bar{\mathbb{F}}_\ell$ by $(*)$. Now we assume $\ell > 2c_n \ell_0^{dDh_K^+ M}$. Then we have

$$\{Dh_K^+ w_1(V)/2, \dots, Dh_K^+ w_n(V)/2\} = \{b'_1 Dh_K^+, \dots, b'_n Dh_K^+\}$$

by Lemma 2.9. Our result comes from this equation. □

Now we can obtain our main results.

Theorem 2.10. *Suppose that w is odd or $w > 2r$. Then the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}_{\mathrm{cycl}}$ is empty under any one of the following situations:*

(a) *w is odd, $\ell \nmid d_K$ and $\ell > C_1$;*

(b) *w is odd, the extension $K/\mathbb{Q}$ has odd degree and $\ell > C_2$;*

(c) *$w > 2r$, $\ell \nmid d_K$ and $\ell > C_1$;*

(d) *$w > 2r$ and $\ell > C_2$;*

(e) *w and n are odd, and $\ell > C_2$.*

Theorem 2.11. *Suppose that w is odd or $w > 2r$. If ℓ does not split in K, then the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$ is empty under any one of the following situations:*

(a) *w is odd, $\ell \nmid d_K$ and $\ell > C'_1$;*

(b) *w is odd, the extension $K/\mathbb{Q}$ has odd degree and $\ell > C'_2$;*

(c) *$w > 2r$, $\ell \nmid d_K$ and $\ell > C'_1$;*

(d) *$w > 2r$ and $\ell > C'_2$;*

(e) *w and n are odd, and $\ell > C'_2$.*

Proofs of Theorem 2.10 and 2.11. We only prove Theorem 2.10 because we can prove Theorem 2.11 by the same way. Suppose that there exists an ℓ-adic Galois representation V which is contained in $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}_{\mathrm{cycl}}$ and take its residual representation $\bar{V}$. If we assume one of the situations (a) and (b) given in Proposition 2.7, then $\bar{V}$ is of uniform tame inertia weight $e_\lambda w/2$ at any place λ of K above ℓ, and thus $e_\lambda w/2$ must be a rational integer. Moreover, by Theorem 1.5, it follows that the tame inertia weight $e_\lambda w/2$ is between 0 and $e_\lambda r$. However, if we assume any one of the conditions (a), (b), (c) and (d), then $e_\lambda w$ is odd for some λ or $e_\lambda w/2 > e_\lambda r$. This is a contradiction. The rest of the assertion related with (e) follows from the fact ([CS], Theorem 1) that the sum of all the tame inertia weights of V at λ must be divisible by e_λ. □

Remark 2.12. To remove the special assumption "ℓ does not split in K" in Theorem 2.11 is impossible in general because there exists such an example, which is pointed out by Akio Tamagawa: Let E be an elliptic curves over K with complex multiplication over K by an imaginary quadratic field $F := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_K(E) \subset K$. Then E is potential everywhere good reduction and thus we may suppose E has everywhere good reduction over K. Put $F_\ell := \mathbb{Q}_\ell \otimes_{\mathbb{Q}} F$, which is a semisimple $\mathbb{Q}_\ell$-algebra. It is well-known that F_ℓ acts faithfully on the Tate-module $V_\ell(E)$ of E and thus $V_\ell(E)$ has a natural structure of 1-dimensional F_ℓ-vector space. If ℓ splits in F, the decomposition $F_\ell \simeq \mathbb{Q}_\ell \times \mathbb{Q}_\ell$ induces a decomposition of $V_\ell(E)$ as a sum of 1-dimensional G_K-stable ℓ-adic representations. For such odd prime ℓ, it is easy to check that $V_\ell(E)$ is an element of the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$, where $\bullet = (2, 2, 1, 1)$.

3 Applications

We give some applications of our results. We use same notation as in the previous section.

3.1 Rasmussen-Tamagawa Conjecture

As a first application, we show a special case of a Conjecture of Rasmussen and Tamagawa. We denote by $\tilde{K}_\ell$ the maximal pro-ℓ extension of $K(\mu_\ell)$ which is unramified away from ℓ.

Definition 3.1. Let $g \geq 0$ be an integer. We denote by $\mathcal{A}(K, g, \ell)$ the set of K-isomorphism classes of abelian varieties A over K, of dimension g, which satisfy the following equivalent conditions:
(1) $K(A[\ell^\infty]) \subset \tilde{K}_\ell$;
(2) The abelian variety A has good reduction outside ℓ and the extension $K(A[\ell])/K(\mu_\ell)$ is an ℓ-extension;
(3) The abelian variety A has good reduction outside ℓ and $A[\ell]$ admits a filtration of G_K-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]$$

such that $\bar{V}_k$ has dimension k for each $1 \leq k \leq 2g$. Furthermore, for each $1 \leq k \leq 2g$, the G_K-action on the space $\bar{V}_k/\bar{V}_{k-1}$ is given by $g.\bar{v} = \chi_\ell(g)^{a_k} \cdot \bar{v}$ for some $a_k \in \mathbb{Z}$.

The equivalently of the above three conditions follows from the criterion of Néron-Ogg-Shafarevich and Lemma 3.4 below (put $G = \mathrm{Gal}(K(A[\ell^\infty])/K)$, $N = \mathrm{Gal}(K(A[\ell^\infty])/K(\mu_\ell))$ and apply Lemma 3.4 to the group $A[\ell]$). The set $\mathcal{A}(K, g, \ell)$ is a finite set because of Faltings' proof of Shafarevich Conjecture. Rasmussen and Tamagawa conjectured that for any ℓ large enough, this set is empty:

Conjecture 3.2 ([RT], Conjecture 1). *The set* $\mathcal{A}(K, g) := \{(A, \ell) \mid [A] \in \mathcal{A}(K, g, \ell),\ \ell : \text{prime number}\}$ *is finite, that is, the set* $\mathcal{A}(K, g, \ell)$ *is empty for any prime* ℓ *large enough.*

We call this conjecture the *Rasmussen-Tamagawa conjecture.* It is known that the Rasmussen-Tamagawa conjecture holds under the following conditions:

(i) $K = \mathbb{Q}$ and $g = 1$ ([RT], Theorem 2);

(ii) K is a quadratic number field other than the imaginary quadratic fields of class number one and $g = 1$ ([RT], Theorem 4).

We consider the semistable reduction case of Conjecture 3.2.

Definition 3.3. (1) We denote by $\mathcal{A}(K, g, \ell)_{\mathrm{st}}$ the set of K-isomorphism classes of abelian varieties in $\mathcal{A}(K, g, \ell)$ with everywhere semistable reduction.
(2) We denote by $\mathcal{A}(K, g, \ell_0, \ell)_{\mathrm{st}}$ the set of K-isomorphism classes of abelian varieties A over K with everywhere semistable reduction, of dimension g, which satisfy the following condition: The abelian variety A has good reduction at some places of K above ℓ_0 and $A[\ell]$ admits a filtration of G_K-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]$$

such that $\bar{V}_k$ has dimension k for each $1 \leq k \leq 2g$.

Clearly, we see $\mathcal{A}(K, g, \ell)_{\mathrm{st}} \subset \mathcal{A}(K, g, \ell_0, \ell)_{\mathrm{st}}$ since $\ell \neq \ell_0$. The set $\mathcal{A}(K, g, \ell)_{\mathrm{st}}$ is finite, however, the set $\mathcal{A}(K, g, \ell_0, \ell)_{\mathrm{st}}$ may be infinite. The Rasmussen-Tamagawa conjecture implies that $\mathcal{A}(K, g, \ell)_{\mathrm{st}}$ will be empty for a prime ℓ large enough. We will prove that $\mathcal{A}(K, g, \ell_0, \ell)_{\mathrm{st}}$ is in fact empty for a prime ℓ large enough which does not split in K. Recall the lemma proved by Rasmussen and Tamagawa (cf. [RT], Lemma 3). Let G be a topological group with a normal pro-ℓ open subgroup N, such that the quotient $\Delta = G/N$ is isomorphic to a subgroup of $\mathbb{F}_\ell^\times$. Because N is pro-ℓ, we see that N has trivial image under any character $\psi \colon G \to \mathbb{F}_\ell^\times$. Hence, there always exists an induced character $\bar{\psi} \colon \Delta \to \mathbb{F}_\ell^\times$. Let $\chi \colon G \to \mathbb{F}_\ell^\times$ be a character such that the induced character $\bar{\chi}$ is an injection $\Delta \hookrightarrow \mathbb{F}_\ell^\times$. Finally, let $\bar{V}$ be a finite dimensional $\mathbb{F}_\ell$-vector space of dimension n on which G acts continuously.

Lemma 3.4. *The vector space V admits a filtration of G_K-modules*

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_n = \bar{V}$$

such that $\bar{V}_k$ has dimension k for each $1 \leq k \leq n$. Furthermore, for each $1 \leq k \leq n$, the G-action on the space $\bar{V}_k/\bar{V}_{k-1}$ is given by $g.\bar{v} = \chi(g)^{a_k} \cdot \bar{v}$ for some $a_k \in \mathbb{Z}$, $0 \leq a_k < \#\Delta$.

Proof. The proof will proceed by the same method as the proof of Lemma 3 of [RT], thus we omit it. □

Take an abelian variety A which is in the set $\mathcal{A}(K, g, \ell)_{\mathrm{st}}$ (resp. $\mathcal{A}(K, g, \ell_0, \ell)_{\mathrm{st}}$). Then $V_\ell(A)$ is an element of the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}_{\mathrm{cycl}}$ (resp. $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$) with $\bullet = (2g, 2, 1, 1)$ (resp. $\bullet = (2g, \ell_0, 1, 1)$) for any $\ell > 2$ (resp. $\ell > \ell_0$). Consequently, we obtain the following results as corollaries of Theorem 2.10 and 2.11:

Corollary 3.5. *The set $\mathcal{A}(K, g, \ell)_{\mathrm{st}}$ is empty under any one of the following situations:*
(a) *$\ell \nmid d_K$ and $\ell > 2^{\delta_1}\binom{2g}{g}$, where $\delta_1 := 2dg + 1$;*
(b) *The extension $K/\mathbb{Q}$ has odd degree and $\ell > 2^{\delta_2}\binom{2g}{g}$, where $\delta_2 := 2d^2g + 1$.*

Corollary 3.6. *Suppose that ℓ does not split in K. The set $\mathcal{A}(K, g, \ell_0, \ell)_{\mathrm{st}}$ is empty under any one of the following situations:*
(a) *$\ell \nmid d_K$ and $\ell > 2\ell_0^{\delta'_1}\binom{2g}{g}$, where $\delta'_1 := 2dgh_K^+$;*
(b) *The extension $K/\mathbb{Q}$ has odd degree and $\ell > 2\ell_0^{\delta'_2}\binom{2g}{g}$, where $\delta'_2 := 2d^2gh_K^+$.*

Remark 3.7. Rasmussen and Tamagawa have shown the finiteness of the set $\mathcal{A}(K, g)_{\mathrm{st}}$ by using the result of [Ra] instead of Theorem 1.5 (unpublished). Our main results in this paper are motivated by their work.

3.2 Irreducibility of ℓ-torsion points of elliptic curves

We consider the following classical question:

Question 3.8. *Does there exist a constant c_K, which depends only on K, such that for any semistable elliptic curve E defined over K without complex multiplication over K, the representation in its ℓ-torsion points $E[\ell]$ is irreducible whenever $\ell > c_K$? Furthermore, if the answer is positive, how can we evaluate such a constant c_K?*

By Mazur's results on a moduli of rational points of modular curve $X_0(N)$ ([Ma]), it is known that $c_{\mathbb{Q}} = 7$. If K is a quadratic field, then the existence of c_K is known and moreover, if the class number of K is 1, then the explicit calculation of c_K is given by Kraus [Kr1]. By combining results on Merel ([Me]) and Momose ([Mo]), Kraus showed the existence of c_K for a number field K which does not contain an imaginary quadratic field of class number 1 ([Kr2]). Moreover, Kraus defined the good condition "(C)" associated with K in *op. cit,* such that the existence and the explicit value of c_K is known if K satisfies this condition.

The following is easy consequence of Corollary 3.6 under the case $g = 1$.

Corollary 3.9. *Let E be an elliptic curve over K with everywhere semistable reduction. Let ℓ_E be the minimal prime number p such that E has good reduction at some finite places of K above p. Suppose ℓ does not split in K. Then $E[\ell]$ is irreducible under any one of the following conditions:*

(a) *$\ell \nmid d_K$ and $\ell > 4\ell_E^{\delta''_1}$, where $\delta''_1 := 2dh_K^+$;*

(b) *The extension $K/\mathbb{Q}$ has odd degree and $\ell > 4\ell_E^{\delta''_2}$, where $\delta''_2 := 2d^2h_K^+$.*

We remark that the above corollary is valid even if E has complex multiplication over K.

3.3 Residual properties of étale cohomology groups

For any semistable elliptic curve E over $\mathbb{Q}$, Serre proved the following ([Se], Section 5.4, Proposition 21, Corollary 1): Let ℓ_E be the minimal prime number p such that E has good reduction at p. Then $E[\ell]$ is irreducible if $\ell > (1 + \ell_E^{1/2})^2$.

As a corollary of Theorem 2.11, we can slightly generalize this fact to étale cohomology groups of odd degree.

Corollary 3.10. *Let X be a proper smooth scheme over K with everywhere semistable reduction and w an odd integer. Let $b_w(X)$ be a w-th Betti number of X and ℓ_X the minimal prime number p such that X has good reduction at some places of K above p. Then there exists a constant C depending only on $b_w(X)$ and ℓ_X such that for any prime number $\ell > C$ which does not split in K, the étale cohomology group $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is not of residually Borel. More precisely, if ℓ does not split in K, $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is not of residually Borel under any one of the following conditions:*

(a) *$\ell \nmid d_K$ and $\ell > 2B_{b_w(X)}\ell_X^{\Delta_1}$, where $\Delta_1 := b_w(X)dh_K^+w$;*

(b) *The extension $K/\mathbb{Q}$ has odd degree and $\ell > 2B_{b_w(X)}\ell_X^{\Delta_2}$, where $\Delta_2 := b_w(X)d^2h_K^+w$.*

Proof. Putting $\bullet := (b_w(X), \ell_X, w, w)$, we see that the dual of $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is contained in the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$. Applying Theorem 2.11, we obtain the desired result. $\square$

For any proper smooth scheme X over K, there exists an finite extension L over K such that $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)$ is everywhere semistable as a representation of G_L for almost all ℓ by Proposition 2.3. If this is the case, we see that $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)^\vee$ satisfies (G-1), (G-2) and (G-3) as a representation of G_L. Thus if we can obtain the explicit description of L, we will able to obtain the analogous result of corollary 3.10 for a prime ℓ large enough which does not split in L. However, it is very difficult to determine such L in general. We can determine this L if X is an abelian variety. If this is the case, Raynaud's criterion of semistable reduction ([Gr], Proposition 4.7) implies that X is everywhere semistable reduction over $L := K(X[3], X[5])$.

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