On congruences of Galois representations of number fields

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Abstract

We give a criterion for two ℓ-adic Galois representations of an algebraic number field to be isomorphic when restricted to a decomposition group, in terms of the global representations mod ℓ. This is applied to prove a generalization of a conjecture of Rasmussen-Tamagawa [14] under a semistablity condition, extending some results [12] of one of the authors. It is also applied to prove a congruence result on the Fourier coefficients of modular forms.

Keywords: ℓ-adic Galois representation, congruence

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1 Introduction

Let K be an algebraic number field (:= finite extension of $\mathbb{Q}$) and let $G_K = \mathrm{Gal}(\bar{K}/K)$ denote its absolute Galois group, where $\bar{K}$ is a fixed algebraic closure of K. Choosing an extension of v to $\bar{K}$, we denote by G_v (resp. I_v) the decomposition (resp. inertia) group of v in G_K. Let E be another algebraic number field, λ a finite place of E of residue characteristic ℓ, and E_λ the completion of E at λ. We denote by $\mathcal{O}_E$ and $\mathcal{O}_{E_\lambda}$ the integer rings of E and E_λ, respectively. Let f_λ denotes the absolute residue degree of λ. We identify any finite place v of an algebraic number field with the corresponding prime ideal, and denote its residue field by k_v and put $q_v := \#k_v$. Throughout the paper, we fix K, E, and a finite place v of K, and let the finite place λ of E of residue characteristic ℓ vary. We denote by ℓ the residue characteristic of λ, and assume $v \nmid \ell$, while u will denote another finite place of K lying above ℓ. All representations of Galois groups denoted V are either $\mathbb{Q}_\ell$- or

E_λ-linear of finite dimension, and assumed to be continuous with respect to the natural topologies. Their "reductions" will be denoted by $\bar{V}$.

In the following, n and e are fixed integers ≥ 1 and e is assumed to be divisible by the absolute ramification index $e(K_u/\mathbb{Q}_\ell)$ of $K_u/\mathbb{Q}_\ell$. For $K, u, v, E, \lambda, n, e$ as above and a real number b, let $\mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, b, e, v)$ denote the set of n-dimensional E_λ-linear representations V of G_K which have the following properties:

– V is semistable at v (in the sense that the action of the inertia is unipotent (including the case where it is trivial)),
– V is E-integral at v in the sense of Definition 2.2,
– V becomes semistable (in the sense of Fontaine [7]) over a finite extension $K'_{u'}$ of K_u whose absolute ramification index $e(K'_{u'}/\mathbb{Q}_\ell)$ divides e,
– V has Hodge-Tate weights $\subset [0, b]$ at u, and
– V is of type (G) in the sense of Definition 2.4,

Our first main result is:

Theorem 1.1. *For any K, E, n, b, v as above, there exists a constant $C = C([E : \mathbb{Q}], n, b, e, q_v)$ such that the following holds: For any prime number $\ell > C$, any places u of K and λ of E both lying above ℓ, and any representations $V \in \mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, b, e, v)$ and $V' \in \mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, (\ell-2)/e^2, e, v)$, if one has $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ both as G_u-representations and G_v-representations, then one has $V \simeq_{\mathrm{ss}} V'$ as G_v-representations. [In particular, if $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ as G_K-representations, then $V \simeq_{\mathrm{ss}} V'$ as G_v-representations.]*

The constant C can be taken explicitly to be

$$C \ := \ \max\{e^2 b + 1, \left(2\binom{n}{[n/2]} q_v^{nb}\right)^{[E:\mathbb{Q}]/f_\lambda}\},$$

where $[x]$ denotes the largest integer not exceeding x.

Here, the meaning of the notations $\equiv_{\mathrm{ss}}$ and $\simeq_{\mathrm{ss}}$ is as follows: we say $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ as G_v-representations if T and T' are G_v-stable $\mathcal{O}_{E_\lambda}$-lattices in V and V', respectively, and the semisimplifications $(T/\lambda T)^{\mathrm{ss}}$ and $(T'/\lambda T')^{\mathrm{ss}}$ are isomorphic as k_λ-linear representations of G_v (this definition does not depend on the choice of the lattices). We say also $V \simeq_{\mathrm{ss}} V'$ as G_v-representations if their semisimplifications are isomorphic as E_λ-linear representations of G_v.

To state a variant of this theorem, let $\mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, b, e, v)'$ be the set of n-dimensional E_λ-linear representations V of G_K which have the following properties:

– V is E-integral at v,
– V becomes semistable over a finite extension $K'_{u'}$ of K_u whose absolute ramification index $e(K'_{u'}/\mathbb{Q}_\ell)$ divides e,
– V has Hodge-Tate weights $\subset [0, b]$ at u, and
– V is of type (G).

Thus $\mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, b, e, v)'$ contains $\mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, b, e, v)$, and the difference is that the elements V of the former are not assumed to be semistable at v. Let $\mathrm{W}_v(V)$ denote the multi-set of Weil weights of V (Def. 2.1) considered as a $\mathbb{Q}_\ell$-linear representation of G_v.

Theorem 1.2. *For K, E, n, b, v as above, the following holds with the same constant $C = C([E : \mathbb{Q}], n, b, e, q_v)$ as in Theorem 1.1: For any prime number $\ell > C$, any places u of K and λ of E both lying above ℓ, and any representations $V \in \mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, b, e, v)'$ and $V' \in \mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, (\ell-2)/e^2, v)'$, if one has $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ both as G_u-representations and G_v-representations, then one has $\mathrm{W}_v(V) = \mathrm{W}_v(V')$. [In particular, if $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ as G_K-representations, then $\mathrm{W}_v(V) = \mathrm{W}_v(V')$.]*

Remark. If we consider representations of type (W) at *all* places $v|q$ for a fixed prime number q and of Hodge-Tate type at *all* places $u|\ell$, we can prove versions of Theorems 1.1 and 1.2 without assuming "type (G)" but with a larger constant

$$C' \ := \ \max\{e^2b+1, \left(2\binom{n}{[n/2]}q^{nb[K:\mathbb{Q}]/[K_v:\mathbb{Q}_q]}\right)^{[E:\mathbb{Q}]/f_\lambda}\}.$$

The proofs are basically the same as in the case of type (G) but use Proposition 2.8 instead of the equality (G) in Definition 2.4.

The constant $C = C([E : \mathbb{Q}], n, b, e, q_v)$ above depends on the coefficient field E. By working mod ℓ rather than mod λ, however, we can suppress this dependence on E as follows:

Theorem 1.3. *For any K, E, n, b, v as above, there exists a constant $\tilde{C} = \tilde{C}(n, b, e, q_v)$ such that the following holds: For any prime number $\ell > C$, any places u of K and λ of E both lying above ℓ, and any representations $V \in \mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, b, e, v)$ and $V' \in \mathrm{Rep}^{(\mathrm{G})}_{E,\lambda,n}(K; u, (\ell-2)/e^2, e, v)$, if one has $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ as G_u-representations and $\det(T - \mathrm{Frob}_v|V) \equiv \det(T - \mathrm{Frob}_v|V') \pmod{\ell\mathcal{O}_E}$, then one has $V \simeq_{\mathrm{ss}} V'$ as G_v-representations. [In* *particular, if* $V \equiv_{\rm ss} V' \pmod{\ell}$ *as* G_K*-representations, then* $V \simeq_{\rm ss} V'$ *as* G'_v*-representations.]*

The constant $\tilde{C}$ *can be taken explicitly to be*

$$\tilde{C} \ := \ \max\{e^2 b + 1, 2\binom{n}{[n/2]} q_v^{nb}\}.$$

After recalling some notions and results on Galois representations in Section 2, we give proofs of the above theorems in Section 3 and several corollaries of Theorem 1.2 in Section 4. In Section 5, we apply Theorem 1.3 with E a Hecke field to prove a congruence result on the Fourier coefficients of modular forms of various levels, where the "independence of E" in the theorem plays a significant role.

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2 Weights

2.1. Weil weights. Let V be a $\mathbb{Q}_\ell$-linear representation of G_v. Choose a lift $\sigma_v \in G_v$ of the q_v-th power Frobenius $\mathrm{Frob}_v \in G_{k_v}$ and let $P(T) = \det(T - \sigma_v|V)$ be the characteristic polynomial of σ_v acting on V. Recall that an algebraic integer α is said to be a q-Weil integer of weight w if $|\iota(\alpha)| = q^{w/2}$ for any field embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, where $|\cdot|$ denotes the absolute value of $\mathbb{C}$.

Definition 2.1. We say that V is of *type (W)* at v if all the roots of $P(T)$ are q_v-Weil integers. If this is the case, we call the weights of the roots of $P(T)$ the *Weil weights* of V at v, and denote by $\mathrm{W}_v(V)$ the multi-set consisting of them.

This definition does not depend on the choice of the Frobenius lift σ_v. Also, the multi-set $\mathrm{W}_v(V)$ is unchanged by a finite extension of the base field K_v.

Now suppose V is an E_λ-linear representation of G_v. The action of the inertia subgroup I_v on V is quasi-unipotent ([22], Appendix); thus there exists a finite extension $K'_{v'}/K_v$ such that the inertia subgroup $I_{v'}$ for $K'_{v'}$ acts unipotently on V (or equivalently, trivially on the semisimplification V^{ss} as an $E_\lambda[G_v]$-module). Hence we can consider the characteristic polynomial $P'(T) = \det(T - \mathrm{Frob}_{v'}|V^{\mathrm{ss}})$ of the Frobenius $\mathrm{Frob}_{v'}$ at v' acting on the E_λ-vector space V^{ss}. (Note that the characteristic polynomial taken with V^{ss} viewed as a $\mathbb{Q}_\ell$-vector space is the product of the "conjugates" of this $P'(T)$.)

Definition 2.2. An E_λ-linear representation V of G_v is said to be *E-integral* at v if, for any finite extension $K'_{v'}/K_v$ for which the inertia action on V is unipotent, the characteristic polynomial $P'(T)$ defined as above has coefficients in $\mathcal{O}_E$.

Note that an E-integral representation of type (W) at v has Weil weights ≥ 0 at v.

For example, if X is a proper smooth variety over K_v, then the $\mathbb{Q}_\ell$-linear dual $V = H^r_{\mathrm{et}}(X_{\bar{K}_v}, \mathbb{Q}_\ell)^*$ of the r-th ℓ-adic étale cohomology group of $X_{\bar{K}_v} := X \otimes_{K_v} \bar{K}_v$ is conjectured to be $\mathbb{Q}$-integral (cf. [18], C_4). This conjecture is known to be true under the assumption of the existence of the Künneth projector ([16], Cor. 0.6 (1)).

We note here that, by the next lemma, there are totally ramified extensions among the finite extensions $K'_{v'}/K_v$ as above (so that, when we want to compare the characteristic polynomials $P'(T)$ for different V's, we can use a $K'_{v'}$ with residue degree $f = 1$):

Lemma 2.3. *If L/K_v is a finite Galois extension, then there exists a totally ramified subextension L'/K_v of L/K_v such that $L = L'L_0$, where L_0 is the maximal unramified subextension of L/K_v.*

Proof. If L/K_v is abelian, this is a consequence of local class field theory. Suppose L/K_v is non-abelian. We proceed by induction on the extension degree $[L : K_v]$. Let σ be a lift in $G := \mathrm{Gal}(L/K_v)$ of the Frobenius in $\mathrm{Gal}(L_0/K_v)$, and set $H := \langle\sigma\rangle$. Then we have $H \subsetneqq G$, and the extension L^H/K_v is a non-trivial totally ramified subextension of L/K_v. Repeating this process with L/K_v replaced by L/L^H, we are reduced to the case of abelian L/K_v. □

2.2. Hodge-Tate weights. Recall that u is a finite place of K lying above ℓ. A $\mathbb{Q}_\ell$-linear representation V of G_u is said (cf. [7]) to be of *Hodge-Tate type* of

Hodge-Tate weights $h_1, ..., h_n$, where $n = \dim_{\mathbb{Q}_\ell}(V)$ and h_i are integers, if one has $V \otimes_{\mathbb{Q}_\ell} \mathbb{C}_\ell \simeq \mathbb{C}_\ell(h_1) \oplus \cdots \oplus \mathbb{C}_\ell(h_n)$ as a $\mathbb{C}_\ell$-semilinear G_u-representation, where $\mathbb{C}_\ell(h)$ denotes the h-th Tate twist of the completion $\mathbb{C}_\ell$ of a fixed algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. If this is the case, let $\mathrm{HT}_u(V)$ denote the multi-set of Hodge-Tate weights of V. Note that $\mathrm{HT}_u(V)$ is unchanged by a finite extension of the base field K_u.

2.3. Tame inertia weights. Let I_u^{tame} the tame inertia group of K at u (= the quotient of the inertia group I_u at u by its maximal pro-ℓ subgroup). A character $\varphi : I_u^{\mathrm{tame}} \to \mathbb{F}_{\ell^h}^\times$ can be written in the form $\varphi = \psi_1^{t_1} \cdots \psi_h^{t_h}$, where ψ_i are the fundamental characters of level h ([19], §1.7) and $0 \leq t_i \leq \ell - 1$. Then we set $\mathrm{TI}_u(\varphi) := \{t_1/e, ..., t_h/e\}$ (as a multi-set), where $e = e(K_u/\mathbb{Q}_\ell)$ is the ramification index of $K/\mathbb{Q}$ at u. Note that, by §1.4 of [19], $\mathrm{TI}_u(\varphi)$ is unchanged by a "moderately" ramified extension of K_u; precisely speaking, if $K'_{u'}/K_u$ is a finite extension of ramification index $e(K'_{u'}/K_u) < (\ell-1)/\max\{t_j |\ 1 \leq j \leq h\}$, then we have $\mathrm{TI}_{u'}(\varphi|_{I_{u'}^{\mathrm{tame}}}) = \mathrm{TI}_u(\varphi)$.

Let V be a $\mathbb{Q}_\ell$-linear representation of G_u, and T a G_u-stable $\mathbb{Z}_\ell$-lattice of V. Set $\bar{T} := T/\ell T$. Then its semisimplification $\bar{T}^{\mathrm{ss}}$ (as an $\mathbb{F}_\ell[G_u]$-module) is tamely ramified (note that its isomorphism class does not depend on the choice of T), and the action of the tame inertia group I_v^{tame} is described by a sum of characters $\varphi_i : I_v^{\mathrm{tame}} \to \mathbb{F}_{\ell^{h_i}}^\times$. Then we define $\mathrm{TI}_u(V)$ (as a multi-set) to be the union of the $\mathrm{TI}_u(\varphi_i)$ for all i.

2.4. Weights of geometric Galois representations. Let V be a $\mathbb{Q}_\ell$-linear representation of G_K. For any multi-set X, we write

$$\Sigma(X) \ := \ \sum_{x \in X} x,$$

whenever the sum on the right-hand side has a meaning.

Definition 2.4. We say that V is of *type (G)* if it is of type (W) at v, of Hodge-Tate type at u, and one has

$$\text{(G)} \qquad \Sigma(\mathrm{W}_v(V)) \ = \ 2\Sigma(\mathrm{HT}_u(V)).$$

If this is the case, we denote this value by $w(V)$ and call it the *total weight* of V.

Note that $\Sigma(\mathrm{W}_v(V))$ and $\Sigma(\mathrm{HT}_u(V))$ are respectively the Weil and Hodge-Tate weights of $\det_{\mathbb{Q}_\ell}(V)$.

Typical examples of V of type (G) include the Tate twists $\mathbb{Q}_\ell(r)$ for $r \in \mathbb{Z}$ and their twists by characters of finite order; their total weights are $2r$.

A priori, the notion of type (G) depends on the places $v \nmid \ell$ and $u \mid \ell$ (so it should be called, say, type ($\mathrm{G}_{u,v}$)), but in practice (i.e., in case V comes from algebraic geometry), it should be independent of the places. The proof of the following proposition, which is modeled on the proof of Lemma 2.1 of [17], has been communicated to us by Yoichi Mieda, to whom we are grateful:

Proposition 2.5. *Let X be a proper smooth variety over K. Let $V = H^r_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)^*$ be the $\mathbb{Q}_\ell$-linear dual of the r-th ℓ-adic étale cohomology group of $X_{\bar{K}} := X \otimes_K \bar{K}$, and put $n = \dim_{\mathbb{Q}_\ell}(V)$. Then we have:*

(i) $\det(V)$ is isomorphic to the twist of $\mathbb{Q}_\ell(nr/2)$ by a character ε of order at most 2. If r is odd, then $\varepsilon = 1$.

(ii) V is of type (G) with respect to any finite places $u \mid \ell$ and $v \nmid \ell$ of K.

Note that, in (i), the Betti number n is even if r is odd by, say, the Hodge symmetry.

Proof. (ii) follows from (i) immediately. To show (i), consider the character $\varepsilon : G_K \to \mathbb{Q}_\ell{}^\times$ defined by $\det(V)(-nr/2)$, where $(-nr/2)$ denotes the $(-nr/2)$-th Tate twist. If v is a finite place of K where X has good reduction, then by [5] V is $\mathbb{Q}$-integral and has all Weil weights equal to r. Hence $\varepsilon(\mathrm{Frob}_v)$ is a Weil integer in $\mathbb{Q}$ of weight 0, i.e., a unit of $\mathbb{Z}$. Since Frob_v's for such v's are dense in G_K, we see that ε takes values in $\mathbb{Z}^\times$. The second statement of (i) follows from Corollary 3.3.5 of [23]. □

In some cases, we can expect the total weight $w(V)$ to be equal also to $2\Sigma(\mathrm{TI}_u(V))$:

Proposition 2.6. *Let V be a $\mathbb{Q}_\ell$-linear semistable representation of G_u with $\mathrm{HT}_u(V) \subset [0, b]$. If $e(K_u/\mathbb{Q}_\ell)b < \ell - 1$, then we have:*

(i) ([3], Thms. 1.0.3 and 1.0.5) $\mathrm{TI}_u(V) \subset [0, b]$.

(ii) ([4], Thm. 1) $\Sigma(\mathrm{HT}_u(V)) = \Sigma(\mathrm{TI}_u(V))$.

The equality (G) holds in general if $K = \mathbb{Q}$:

Lemma 2.7. *Let q be a prime number $\neq \ell$. If V is a $\mathbb{Q}_\ell$-linear representation of $G_{\mathbb{Q}}$ which is of type (W) at q and of Hodge-Tate type at ℓ, then V is of type (G).*

Proof. By taking the determinant, we are reduced to the case $\dim_{\mathbb{Q}_\ell}(V) = 1$. Then V is geometric (in the sense of Fontaine-Mazur [9] (note that a one-dimensional $\mathbb{Q}_\ell$-representation is de Rham if and only if it is Hodge-Tate) and hence is a twist by a finite character of $\mathbb{Q}_\ell(r)$ for some integer r. Thus (G) holds for V. $\square$

If $K \neq \mathbb{Q}$, the equality (G) may not hold even for a geometric representation. For example, let K be an imaginary quadratic field, E an elliptic curve over K such that $\mathrm{End}_K(E) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq K$, and ℓ a prime number which splits in K as $\ell = \lambda\lambda'$. Let V be a one dimensional G_K-subrepresentation of the ℓ-adic Tate module $T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ of E. Then V is of type (W) of Weil weight 1 at any $v \nmid \ell$, while it is of Hodge-Tate type of Hodge-Tate weight 0 or 1 at λ.

If we do not assume the equality (G), we can in fact prove an equality which is fairly close to (G) under a mild condition:

Proposition 2.8. *Let V be a $\mathbb{Q}_\ell$-linear representation of G_K and q a prime number $\neq \ell$. Assume V is of type (W) at all places $v|q$ and of Hodge-Tate type at all places $u|\ell$. Then we have*

$$\sum_{v|q} [K_v : \mathbb{Q}_q]\Sigma(\mathrm{W}_v(V)) \ = \ 2\sum_{u|\ell} [K_u : \mathbb{Q}_\ell]\Sigma(\mathrm{HT}_u(V)).$$

Proof. The induced representation $\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(V)$ is a representation of $G_{\mathbb{Q}}$ which is of type (W) at q and of Hodge-Tate type at ℓ, and hence we have

$$\Sigma(\mathrm{W}_q(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(V))) \ = \ 2\Sigma(\mathrm{HT}_\ell(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(V)))$$

by Lemma 2.7. We then observe that

$$\mathrm{W}_q(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(V)) \ = \ \coprod_{v|q} [K_v : \mathbb{Q}_q]\mathrm{W}_v(V),$$
$$\mathrm{HT}_\ell(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(V)) \ = \ \coprod_{u|\ell} [K_u : \mathbb{Q}_\ell]\mathrm{HT}_\ell(V),$$

where the multiple mX of a multi-set X by a positive integer m is defined in the obvious manner. Indeed, we have

$$(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(V))|_{G_q} \ = \ \bigoplus_{v|q} \mathrm{Ind}_{G_v}^{G_q}(V|_{G_v})$$

by Mackey's formula ([21], Section 7.3, Proposition 22), and

$$\mathrm{W}_q(\mathrm{Ind}_{G_v}^{G_q}(V|_{G_v})) \;=\; [K_v:\mathbb{Q}_q]\mathrm{W}_v(V|_{G_v})$$

by definition of the induced representation and by the invariance of the Weil weights by finite extensions of the base field. Similar equalities hold for $u|\ell$ and $\mathrm{Ind}_{G_u}^{G_\ell}(V|_{G_u})$. □

3 Proof of the theorems

We begin with a version of the gap principle:

Lemma 3.1. *Let E, n, v be as before, and let $w \in \mathbb{R}_{\geq 0}$ be given. Then there exists a constant $C_1 = C_1([E:\mathbb{Q}], n, q_v^w) > 0$ such that, for any prime $\ell > C_1$ and for any n-dimensional E_λ-linear representations V, V' of G_v which are of type (W), E-integral at v and such that $\Sigma(\mathrm{W}_v(V)), \Sigma(\mathrm{W}_v(V'))$ are in $[0, [E_\lambda:\mathbb{Q}_\ell]\cdot w]$, the following (i) and (ii) hold:*
(i) If $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ as G_v-representations, then $\mathrm{W}_v(V) = \mathrm{W}_v(V')$.
(ii) Assume further that V^{ss} and $(V')^{\mathrm{ss}}$ are unramified. If $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ as G_v-representations, then $V \simeq_{\mathrm{ss}} V'$ as G_v-representations.

The constant C_1 can be taken explicitly to be

$$C_1 := \left(2\binom{n}{[n/2]}q_v^{w/2}\right)^{[E:\mathbb{Q}]/f_\lambda}.$$

We have also the following mod ℓ version of (ii) above, in which the constant is independent of $[E:\mathbb{Q}]$:

Lemma 3.2. *Let E, n, v be as before, and let $w \in \mathbb{R}_{\geq 0}$ be given. Then there exists a constant $\tilde{C}_1 = \tilde{C}_1(n, q_v^w) > 0$ such that, for any prime $\ell > C_1$ and for any n-dimensional E_λ-linear representations V, V' of G_v such that $V^{\mathrm{ss}}, (V')^{\mathrm{ss}}$ are unramified and which are of type (W), E-integral at v and such that $\Sigma(\mathrm{W}_v(V)), \Sigma(\mathrm{W}_v(V'))$ are in $[0, [E_\lambda:\mathbb{Q}_\ell]\cdot w]$, the following holds: If $\det(T - \mathrm{Frob}_v|V) \equiv \det(T - \mathrm{Frob}_v|V') \pmod{\ell\mathcal{O}_E})$, then one has $V \simeq_{\mathrm{ss}} V'$ as G_v-representations.*

The constant $\tilde{C}_1$ can be taken explicitly to be

$$\tilde{C}_1 := 2\binom{n}{[n/2]}q_v^{w/2}.$$

Proof. As the proofs are similar, we only give a proof of Lemma 3.1. Choose a totally ramified extension $K'_{v'}/K_v$ over which V and V' become semistable (cf. Lem. 2.3). Let $P(T) = \det(T - \mathrm{Frob}_{v'}|\ V^{\mathrm{ss}})$ and $P'(T) = \det(T - \mathrm{Frob}_{v'}|\ (V')^{\mathrm{ss}})$ be the characteristic polynomials (taken as E_λ-linear representations) of the Frobenius $\mathrm{Frob}_{v'}$ at v' acting on the semisimplifications V^{ss} and $(V')^{\mathrm{ss}}$, respectively. By assumption, they have coefficients in $\mathcal{O}_E$. By assumption on the weights, for any embedding $E \hookrightarrow \mathbb{C}$, the terms of T^{n-i} have coefficients of absolute value $\leq \binom{n}{i} q_v^{w/2}$ Note that $\Sigma(\mathrm{W}_v(V))$ is the sum of the Weil weights of V as a $\mathbb{Q}_\ell$-linear representation, and hence the sum of the Weil weights of the roots of $P(T)$ is in $[0, w]$). Set $C_1 := (2 \max_{0 \leq i \leq n} \binom{n}{i} q_v^{w/2})^{[E:\mathbb{Q}]/f_\lambda} = (2\binom{n}{[n/2]} q_v^{w/2})^{[E:\mathbb{Q}]/f_\lambda}$. Then if $\ell > C_1$, we have

$$
\begin{aligned}
& V \ \equiv_{\mathrm{ss}} \ V' \pmod{\lambda} \quad \text{as } G_v\text{-representations} \\
\iff & P(T) \ \equiv \ P'(T) \pmod{\lambda} \\
\iff & P(T) \ = \ P'(T).
\end{aligned}
$$

Here, the last equivalence follows from the next lemma. This implies that $\mathrm{W}_v(V) = \mathrm{W}_v(V')$. If V^{ss} and $(V')^{\mathrm{ss}}$ are unramified, then they are determined by the actions of Frob_v, and hence the equality $P(T) = P'(T)$ is equivalent to $V \simeq_{\mathrm{ss}} V'$. □

Lemma 3.3. *Let a be a non-zero integer of E and C_0 a real number > 0. If $a \equiv 0 \pmod{\lambda}$ (resp. $a \equiv 0 \pmod{\ell\mathcal{O}_E}$) and $|\iota(a)| \leq C_0$ for any embedding $\iota : E \hookrightarrow \mathbb{C}$, then we have $\ell \leq C_0^{[E:\mathbb{Q}]/f_\lambda}$ (resp. $\ell \leq C_0$).*

Proof. If $\lambda | a$ (resp. $\ell | a$ in $\mathcal{O}_E$), then by taking the norm $N : E^\times \to \mathbb{Q}^\times$, we have $\ell^{f_\lambda} \leq |N(a)|$ (resp. $\ell^{[E:\mathbb{Q}]} \leq |N(a)|$). If $|\iota(a)| \leq C_0$, then by taking the norm (or product over all ι), we have $|N(a)| \leq C_0^{[E:\mathbb{Q}]}$. The required inequality follows from these two inequalities. □

We need one more lemma:

Lemma 3.4. *Let G be a profinite group and T, T' be free $\mathcal{O}_{E_\lambda}$-modules on which G acts continuously and $\mathcal{O}_{E_\lambda}$-linearly. Let $(T/\lambda T)^{\mathrm{ss}}$ and $(T/\ell T)^{\mathrm{ss}}$ be the semisimplifications of $T/\lambda T$ and $T/\ell T$ as $k_\lambda[G]$-modules, respectively. Let e be the ramification index of $E_\lambda/\mathbb{Q}_\ell$. Then we have:*

(i) $(T/\ell T)^{\mathrm{ss}}$ is isomorphic to the direct-sum of e copies of $(T/\lambda T)^{\mathrm{ss}}$.

(ii) If $(T/\lambda T)^{\mathrm{ss}} \simeq (T'/\lambda T')^{\mathrm{ss}}$, then $(T/\ell T)^{\mathrm{ss}} \simeq (T'/\ell T')^{\mathrm{ss}}$.

Proof. Part (ii) follows from Part (i) immediately. To prove (i), consider the filtration

$$T/\ell T = T/\lambda^e T \supset \lambda T/\lambda^e T \supset \cdots \supset \lambda^e T/\lambda^e T = 0.$$

Then "multiplication by λ" (where λ is identified with a uniformizer at λ) induces isomorphisms $\lambda^i T/\lambda^{i+1}T \to \lambda^{i+1}T/\lambda^{i+2}T$ of the graded quotients as $k_\lambda[G]$-modules. It then follows that $(T/\ell T)^{\mathrm{ss}} \simeq ((T/\lambda T)^{\mathrm{ss}})^{\oplus e}$. □

Now we can prove the theorems. We only prove Theorem 1.1 and 1.2, the proof of Theorem 1.3 being similar. Let $C = \max\{e^2b+1, (2\binom{n}{[n/2]}q_v^{nb})^{[E:\mathbb{Q}]/f_\lambda}\}$, as in Theorem 1.1. Choose a finite totally ramified extension $K'_{u'}/K_u$, with absolute ramification index e^2, over which V and V' become semistable (cf. Lem. 2.3). If $\ell > C$, then $e^2b < \ell - 1$. Take K' a finite extension of K and $u'|u$ a place of K' such that the completion of K' at u' is $K'_{u'}$. By assumption, we have $\mathrm{HT}_{u'}(V) \subset [0, b]$. Then by (i) of the Proposition 2.6, we have $\mathrm{TI}_{u'}(V) \subset [0, b]$. The same holds for V', since we have $\mathrm{TI}_{u'}(V) = \mathrm{TI}_{u'}(V')$ by the assumption $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ as G_u-representations (Note that, by Lemma 3.4, we have also $V \equiv_{\mathrm{ss}} V' \pmod{\ell}$ as $\mathbb{F}_\ell[G_u]$-modules, where V and V' are now regarded as $\mathbb{Q}_\ell$-linear representations, so that the definition of TI_u and Proposition 2.6 are applicable). Now we recall that V and V' are of type (G). By (ii) of Proposition 2.6, we have $\Sigma(\mathrm{TI}_{u'}(V)) = \Sigma(\mathrm{HT}_{u'}(V)) = \Sigma(\mathrm{HT}_u(V)) = (1/2)\Sigma(\mathrm{W}_v(V))$, and these are also equal to $\Sigma(\mathrm{TI}_{u'}(V')) = \Sigma(\mathrm{HT}_{u'}(V')) = \Sigma(\mathrm{HT}_u(V')) = (1/2)\Sigma(\mathrm{W}_v(V'))$. Since $\mathrm{HT}_u(V) \subset [0, b]$, these are bounded by $[E_\lambda : \mathbb{Q}_\ell]\cdot nb$. In particular, total weights $\Sigma(\mathrm{W}_v(V))$ and $\Sigma(\mathrm{W}_v(V'))$ are $\leq [E_\lambda : \mathbb{Q}_\ell]\cdot 2nb$. By (i) (resp. (ii)) of Lemma 3.1, the assumption that $V \equiv_{\mathrm{ss}} V' \pmod{\lambda}$ as G_v-representations implies that $\mathrm{W}_v(V) = \mathrm{W}_v(V')$ (resp. $V \simeq_{\mathrm{ss}} V'$ as G_v-representations) if $\ell > (2\binom{n}{[n/2]}q_v^{nb})^{[E:\mathbb{Q}]/f_\lambda}$. □

4 Corollaries

Here we give several corollaries of Theorem 1.2, which are motivated by a conjecture of Rasmussen and Tamagawa ([14]; see also [2], [12], [13] and [15]). The notations (K, E, n, b, e, v, u, ℓ, λ, $C = C([E:\mathbb{Q}], n, b, e, q_v)$, ...) are the same as in the theorem. In this section, $V = V_X^r$ will be the E_λ-linear dual $H^r_{\mathrm{et}}(X_{\bar{K}}, E_\lambda)^*$ of the r-th λ-adic étale cohomology group, where X is a smooth proper variety (variety := separated scheme of finite type over a field) over K

and $X_{\bar{K}}$ denotes its base extension to $\bar{K}$. We set $\bar{V} = \bar{V}_X^r := T/\lambda T$, choosing a G_K-stable $\mathcal{O}_{E_\lambda}$-lattice in V, and let $\bar{V}^{\mathrm{ss}} = \bar{V}_X^{r,\mathrm{ss}}$ be its semisimplification as a $k_\lambda[G_K]$-module ($\bar{V}_X^{r,\mathrm{ss}}$ does not depend on the choice of T). To state the first corollary, we make the following hypothesis on $\bar{V}^{\mathrm{ss}}$:

Hypothesis (H). Each simple factor $\bar{W}$ of $\bar{V}^{\mathrm{ss}}$ lifts to an E_λ-linear representation W of G_K of the form $H^s_{\mathrm{et}}(Y_{\bar{K}}, E_\lambda)^*$ which is semistable at all $u \mid \ell$ and $\mathrm{HT}_u(W) \subset [0, \ell - 2]$, where Y is a proper smooth variety over K and s is some non-negative integer.

Corollary 4.1. *For any prime $\ell > C$, any* odd *integer r with $1 \le r \le b$, any places u of K and λ of E both lying above ℓ, and any smooth proper variety X which has the r-th Betti number $\le n$, has potentially good reduction at v, and has semistable reduction at some place $u \mid \ell$, if (H) is true for $\bar{V}_X^{r,\mathrm{ss}}$, then none of the simple factors of $\bar{V}_X^{r,\mathrm{ss}}$ are of odd dimension.*

Proof. Note first that, if s is odd, then $H^s_{\mathrm{et}}(Y_{\bar{K}}, E_\lambda)$ has even dimension by (GAGA and) Hodge theory. Now, let $\bar{W}_1, ..., \bar{W}_k$ be the simple factors of $\bar{V}^{\mathrm{ss}}$. By (H), each $\bar{W}_i$ lifts to a geometric W_i with $\mathrm{HT}_u(W_i) \subset [0, \ell - 2]$. If one of the W_i has odd dimension, then it must have even weight, while V has odd weight r, since X has potentially good reduction at v. Thus the corollary follows from Theorem 1.2 by putting $V' := W_1 \oplus \cdots \oplus W_k$. □

As a special case where the Hypothesis (H) holds, we have:

Corollary 4.2. *For any prime number $\ell > C$, any* odd *integer r with $1 \le r \le b$, any places u of K and λ of E both lying above ℓ, and any smooth proper variety X over K which has r-th Betti number $\le n$, has potentially good reduction at v, and has semistable reduction at u, the Galois representation on $\bar{V}_X^{r,\mathrm{ss}}$ is not the sum of integral powers mod ℓ cyclotomic characters.*

In fact, we can generalize this a bit as follows. Let χ and $\bar{\chi}$ denote respectively the ℓ-adic and mod ℓ cyclotomic characters of G_K.

Corollary 4.3. *Assume E contains the e^2-th roots of unity. Then for any prime number $\ell > C$ such that $\ell \equiv 1 \pmod{e^2}$, any* odd *integer r with $1 \le r \le b$, any places u of K and λ of E both lying above ℓ, and any smooth proper variety X over K which has r-th Betti number $\le n$, has potentially good reduction at v, and acquires semistable reduction over a finite extension $K'_{u'}/K_u$ with absolute ramification index $e(K'_{u'}/\mathbb{Q}_\ell)$ dividing e, the Galois representation $\bar{V}_X^{r,\mathrm{ss}}$ is not the sum of characters of G_K of the form $\bar{\varepsilon}_i \bar{\chi}^{b_i}$,* *where $\bar{\varepsilon}_i : G_K \to k_\lambda{}^\times$ are characters unramified at u and of finite order dividing the order of the group of roots of unity in E, and b_i are integers.*

Proof. Suppose X has semistable reduction over $K'_{u'}$ with $e(K'_{u'}/\mathbb{Q}_\ell) \mid e$. We may assume $e(K'_{u'}/\mathbb{Q}_\ell) = e$. Suppose $\bar{V}^{\mathrm{ss}}$ is the sum of the characters $\bar{\varepsilon}_i\bar{\chi}^{b_i}$ as above. Then the action of the tame inertia group $I_{u'}^{\mathrm{tame}}$ at u' on the i-th factor is via $\bar{\chi}^{b_i}$, which equals θ^{eb_i}, where θ is the fundamental character of $I_{u'}^{\mathrm{tame}}$ of level 1 ([19], Sect. 1.8, Prop. 8). By (i) of Proposition 2.6, we have $eb_i \equiv c_i \pmod{\ell - 1}$ with $0 \le c_i \le eb$. Since $e^2 \mid \ell - 1$, we have $b_i = b_{0i} + \frac{\ell-1}{e^2} j$ with $0 \le b_{0i} \le b$ and $0 \le j < e^2$. Set $\bar{\kappa} := \bar{\chi}^{(\ell-1)/e^2}$ and let $\kappa : G_K \to E_\lambda{}^\times$ be its Teichmüller lift. Since the e^2-th power of κ is trivial, it takes values in $E^\times$. Similarly, the Teichmüller lift ε_i of $\bar{\varepsilon}_i$ has also values in $E^\times$. Now each character $\bar{\varepsilon}_i\bar{\chi}^{b_i} = \bar{\varepsilon}_i\bar{\kappa}^j\bar{\chi}^{b_{0i}}$ lifts to the character $\varepsilon_i\kappa^j\chi^{b_{0i}} : G_K \to E_\lambda{}^\times$, or to the 1-dimensional E_λ-linear E-integral geometric representation $E_\lambda(\varepsilon_i\kappa^j) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(b_{0i})$, where $E_\lambda(\varepsilon_i\kappa^j)$ is the twist of the trivial representation E_λ by the finite character $\varepsilon_i\kappa^j$ and $\mathbb{Q}_\ell(b_{0i})$ denotes the b_{0i}-th Tate twist. Let V' be the direct-sum of these representations. By Theorem 1.2, we have $\mathrm{W}_v(V) = \mathrm{W}_v(V')$, but $\mathrm{W}_v(V) = \{r, ..., r\}$ (since X has potentially good reduction at v) while $\mathrm{W}_v(V') = \{2b_{01}, ..., 2b_{0n}\}$, which is a contradiction if r is odd. ☐

Specializing further, we have:

Corollary 4.4. *Let $K = \mathbb{Q}$. Assume E contains the e^2-th roots of unity. Then for any prime number $\ell > C$ such that $\ell \equiv 1 \pmod{e^2}$, for any* odd *integer r with $1 \le r \le b$, and for any smooth proper variety X over $\mathbb{Q}$ which has r-th Betti number $\le n$, has good reduction outside ℓ and acquires semistable reduction over a finite extension $K'_{u'}/\mathbb{Q}_\ell$ with absolute ramification index $e(K'_{u'}/\mathbb{Q}_\ell)$ dividing e, the Galois representation on $\bar{V}$ is not Borel.*

Here, we say that the representation $\bar{V}$ is Borel if the action of $G_\mathbb{Q}$ is given by upper-triangular matrices with respect to a suitable k_λ-basis of $\bar{V}$.

Proof. Indeed, if it is Borel, its semisimplification is a sum of characters, which are unramified outside ℓ by assumption. Since the base field is $\mathbb{Q}$, they are powers of the mod ℓ cyclotomic character. Now the the result follows from the previous corollary. ☐

5 Congruences of modular forms

We use the same notations as in the Introduction, except that we always suppose $K = \mathbb{Q}$ and write q for q_v in this section. We put $\varphi(N) = \#(\mathbb{Z}/N\mathbb{Z})^\times$ for any positive integer N and denote by $\bar{\mathbb{Z}}$ the integer ring of $\bar{\mathbb{Q}}$. The goal of this section is to give a proof of the following congruence result on the Fourier coefficients of modular forms. For any integers $k, N \geq 1$ and a character $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, let $S_k(N, \epsilon)$ denote the $\mathbb{C}$-vector space of cusp forms of weight k, level N and Nebentypus character ϵ. For a normalized Hecke eigenform $f(z) = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z} \in S_k(N, \epsilon)$, integers i, j and a prime number ℓ, consider the following condition on the Fourier coefficients $a_p(f)$ of f:

$(\mathrm{C}_{i,j:\ell})$ $a_p(f) \equiv p^i + p^j \pmod{\ell\bar{\mathbb{Z}}}$ for all but finitely many primes $p \nmid \ell N$.

For fixed k and N, it is well known (cf. e.g. Thm. 10 of [20] and the Introduction of [11]) that there are only finitely many exceptional primes, and a fortiori finitely many primes ℓ for which $(\mathrm{C}_{i,j:\ell})$ hold for some i, j and $f \in S_k(N, \epsilon)$. Until recently, however, the situation had not been very clear when we let k and N vary; as for recent works, see [10] for the case of modular Abelian varieties and [1] for the case of modular forms on $\Gamma_0(N)$. In this vein, we show the following by using Theorem 1.3:

Theorem 5.1. *Fix a prime number q. For any integer $k \geq 1$, any prime $\ell > 4q^{2(k-1)}$, any integer N such that $q \nmid N$, $\ell \nmid \varphi(N)$ and $\ell^2 \nmid N$, any character $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, and any normalized Hecke eigenform $f \in S_k(N, \epsilon)$, we have the following:*

(i) The condition $(\mathrm{C}_{i,j:\ell})$ can hold only if $i \equiv j \equiv (k-1)/2 \pmod{\ell - 1}$.

(ii) The condition $(\mathrm{C}_{i,j:\ell})$ holds for no i and j if either $k = 1$, k is even, or $\ell \nmid N$.

We begin by proving a lemma. For any f as in the theorem, we denote by $E = \mathbb{Q}_f$ the field obtained by adjoining all Fourier coefficients of f to $\mathbb{Q}$, which is a finite extension of $\mathbb{Q}$. We regard ϵ as a character with values in ${\mathcal{O}_E}^\times$. Denote by $\bar{\epsilon}$ (resp. $\bar{\epsilon}_\lambda$) the composite $(\mathbb{Z}/N\mathbb{Z})^\times \stackrel{\epsilon}{\to} {\mathcal{O}_E}^\times \stackrel{\mathrm{mod}\ \ell}{\to} (\mathcal{O}_E/\ell\mathcal{O}_E)^\times$ (resp. $(\mathbb{Z}/N\mathbb{Z})^\times \stackrel{\epsilon}{\to} {\mathcal{O}_E}^\times \stackrel{\mathrm{mod}\ \lambda}{\to} (\mathcal{O}_E/\lambda\mathcal{O}_E)^\times$). Let

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_{E_\lambda}(V_{f,\lambda})$$

be the 2-dimensional E_λ-linear representation of $G_{\mathbb{Q}}$ associated with f. Thus if $p \nmid \ell N$, then $V_{f,\lambda}$ is unramified at p and one has

$$\det(T - \mathrm{Frob}_p | V_{f,\lambda}) \;=\; T^2 - a_p(f)T + \epsilon(p)p^{k-1}.$$

In particular, it is E-integral at p in the sense of Definition 2.2. One has $\mathrm{W}_p(V_{f,\lambda}) = \{(k-1)/2, (k-1)/2\}$. It is crystalline (resp. semistable) at ℓ if $\ell \nmid N$ (resp. $\ell^2 \nmid N$).

Lemma 5.2. *Suppose $\ell > 2$. Let $k \geq 1$ and $N \geq 1$ be integers with $\ell \nmid \varphi(N)$. Let $\epsilon\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a character. Suppose that a normalized Hecke eigenform $f \in S_k(N, \epsilon)$ satisfies the condition $(\mathrm{C}_{i,j:\ell})$ for some i, j. Then $\bar{\epsilon}$ has values in fact in the canonical image of $\mathbb{F}_\ell^\times$ in $(\mathcal{O}_E/\ell\mathcal{O}_E)^\times$. Moreover, the following holds:*

(i) We have $\bar{\epsilon}(x \pmod N) = x^{i+j-(k-1)} \pmod \ell$ for any x prime to N.

(ii) If $\ell \nmid N$, then we have $i + j \equiv k - 1 \pmod{\ell - 1}$ and $\bar{\epsilon} = 1$.

Proof. By assumption, we have $\mathrm{Tr}(\mathrm{Frob}_p | V_{f,\lambda}) \equiv p^i + p^j \pmod{\ell\mathcal{O}_E}$ for all but finitely many $p \nmid \ell N$. In particular, we have

$$\rho_{f,\lambda} \;\equiv_{\mathrm{ss}}\; \chi^i \oplus \chi^j \pmod \lambda \tag{1}$$

as k_λ-linear representations of $G_{\mathbb{Q}}$ (This holds because $\ell > \dim \rho_{f,\lambda}$; see e.g. Lemma 2.10 of [13]), and then we have also $\epsilon(p)p^{k-1} \equiv p^{i+j} \pmod \lambda$. Hence we see that

$$\bar{\epsilon}_\lambda(x \pmod N) \;=\; x^{i+j-(k-1)} \pmod \lambda \tag{2}$$

for any $\lambda | \ell$ and any integer x prime to N.

(i) Since the kernel of the projection $(\mathcal{O}_E/\ell\mathcal{O}_E)^\times \to \prod_{\lambda|\ell} (\mathcal{O}_E/\lambda\mathcal{O}_E)^\times$ has ℓ-power order, if $\ell \nmid \varphi(N)$, then the homomorphism $\prod_{\lambda|\ell} \bar{\epsilon}_\lambda : (\mathbb{Z}/N\mathbb{Z})^\times \to \prod_{\lambda|\ell} (\mathcal{O}_E/\lambda\mathcal{O}_E)^\times$ lifts uniquely to a homomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathcal{O}_E/\ell\mathcal{O}_E)^\times$, which is $\bar{\epsilon}$. According to (2), it is given by

$$\bar{\epsilon}(x \pmod N) \;=\; x^{i+j-(k-1)} \pmod{\ell\mathcal{O}_E} \tag{3}$$

for any integer x prime to N.

(ii) Suppose $\ell \nmid N$. Then (3) must hold for $x = \ell$, which is possible only if $i + j \equiv k - 1 \pmod{\ell - 1}$. In particular, we obtain $\bar{\epsilon} = 1$. $\square$

Proof of Theorem 5.1. (i) Suppose $\ell \nmid \varphi(N)$ and $\ell^2 \nmid N$. Then $\rho_{f,\lambda}$ is semistable at ℓ. By assumption, we have $\mathrm{Tr}(\mathrm{Frob}_q|V_{f,\lambda}) \equiv q^i + q^j \pmod{\ell\mathcal{O}_E}$. Combining this with Lemma 5.2 (i), we obtain $\det(T - \mathrm{Frob}_q|V_{f,\lambda}) \equiv \det(T - \mathrm{Frob}_q|\chi^i \oplus \chi^j) \pmod{\ell\mathcal{O}_E}$. We also have the congruence (1). Therefore, if $\ell > 4q^{2(k-1)}$, it follows from Theorem 1.3 (applied with $V' = \chi^{i'} \oplus \chi^{j'}$, where i', j' are integers in $[0, \ell-2]$ such that $i' \equiv i$, $j' \equiv j \pmod{\ell - 1}$) that $\rho_{f,\lambda} \simeq_{\mathrm{ss}} \chi^i \oplus \chi^j$ as E_λ-linear representations of the decomposition group G_q of q. Looking at the Weil weights, we obtain $i \equiv j \equiv (k-1)/2 \pmod{\ell - 1}$.

(ii) If k is even, then the impossibility of $(\mathrm{C}_{i,j:\ell})$ follows from Part (i).

If $k = 1$ and the congruence condition $(\mathrm{C}_{i,j:\ell})$ holds, then Part (i) together with (1) implies that $\bar{\rho}_{f,\lambda} := \rho_{f,\lambda} \pmod{\lambda}$ is unipotent and, in particular, $\mathrm{Im}(\bar{\rho}_{f,\lambda})$ is an ℓ-group. On the other hand, if $k = 1$, then by [6], $\mathrm{Im}(\rho_{f,\lambda})$ is finite and its image in $\mathrm{PGL}_2(\mathcal{O}_{E_\lambda})$ is either dihedral, A_4, S_4 or A_5. Since the kernel of the reduction map $\mathrm{GL}_2(\mathcal{O}_{E_\lambda}) \to \mathrm{GL}_2(k_\lambda)$ is pro-ℓ, the representation $\bar{\rho}_{f,\lambda}$ cannot be unipotent if $\ell \geq 3$.

Finally, assume $\ell \nmid N$. Then $\rho_{f,\lambda}$ is crystalline at ℓ, and thus the Fontaine-Laffaille theory [8] implies that the tame inertia weights and the Hodge-Tate weights of $\rho_{f,\lambda}$ coincide with each other. Hence it follows from (1) that $\{i, j\} \equiv \{0, k-1\} \pmod{\ell - 1}$. Since $\ell > k$, we obtain $\{(k-1)/2, (k-1)/2\} = \{0, k-1\}$, which is impossible unless $k = 1$. □

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