NON-EXISTENCE OF CERTAIN CM ABELIAN VARIETIES WITH PRIME POWER TORSION

YOSHIYASU OZEKI*

Abstract. In this paper, we study a conjecture of Rasmussen and Tamagawa, on the finiteness of the set of isomorphism classes of abelian varieties with constrained prime power torsion. Our result is related with abelian varieties which have complex multiplication over their fields of definition.

1. Introduction

Let K be a finite extension of $\mathbb{Q}$, $\bar{K}$ an algebraic closure of K and $G_K = \mathrm{Gal}(\bar{K}/K)$ the absolute Galois group of K. For a prime number ℓ, we denote by $K(\mu_\ell)$ the field generated over K by the ℓ-th roots of unity. If A is an abelian variety over K, denote by $K(A[\ell])$ the field generated by K and the coordinates of all ℓ-torsion points of A. We denote by $\mathcal{A}(K,g,\ell)$ the set of K-isomorphism classes of g-dimensional abelian varieties A over K which satisfy the following conditions.

(RT_ℓ) $K(A[\ell])$ is an ℓ-extension of $K(\mu_\ell)$, that is, $[K(A[\ell]) : K(\mu_\ell)] = \ell^a$ for some integer $a \geq 0$.

($\mathrm{RT}_{\mathrm{red}}$) The abelian variety A has good reduction away from ℓ over K.

It follows from the condition ($\mathrm{RT}_{\mathrm{red}}$) and Faltings' result on the Shafarevich Conjecture that $\mathcal{A}(K,g,\ell)$ is a finite set. Rasmussen and Tamagawa suggested that such finiteness should hold if we take the union of these sets for ℓ varying over all primes.

Conjecture 1.1 ([18, Conjecture 1]). *The set* $\mathcal{A}(K,g) := \{([A],\ell); [A] \in \mathcal{A}(K,g,\ell),\ \ell :$ prime number$\}$ *is finite, that is, the set* $\mathcal{A}(K,g,\ell)$ *is empty for any prime number* ℓ *large enough.*

This conjecture is proved only in a few cases. For example, Conjecture 1.1 in the case where K is the rational number field or a certain quadratic field, with $g = 1$ is proved by Rasmussen and Tamagawa in [18]. Their proof is based on results on K-rational points on modular curves in [14] and [15]. Arguments for algebraic points on Shimura curves (cf. [2], [3]) also give results on Conjecture 1.1 for QM-abelian surfaces over certain quadratic fields K.

2000 *Mathematics Subject Classification.* Primary 11G10; Secondary 14K22.

Key words and phrases. Abelian varieties, compatible systems.

*Partly supported by the Grant-in-Aid for Research Activity Start-up, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

In this paper, we prove Conjecture 1.1 for abelian varieties in $\mathcal{A}(K,g,\ell)$ which satisfy the condition that the representations associated with their ℓ-adic Tate modules are abelian; we are interested in the subset $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ of $\mathcal{A}(K,g,\ell)$ consisting of K-isomorphism classes of g-dimensional abelian varieties A over K which satisfy the additional condition:

($\mathrm{RT}_{\mathrm{ab}}$) The representation $\rho_{A,\ell}\colon G_K \to GL(T_\ell(A))$ associated with the ℓ-adic Tate module $T_\ell(A)$ of A has an abelian image.

In fact, we prove a more general result as follows. Denote by $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ the set of K-isomorphism classes of g-dimensional abelian varieties A over K which satisfy ($\mathrm{RT}_{\mathrm{ab}}$) and the condition:

$(\mathrm{RT}_\ell)'$ For some finite extension L of K which is unramified at all places of K above ℓ, $L(A[\ell])$ is an ℓ-extension of $L(\mu_\ell)$.

Remark that, differently from elements in $\mathcal{A}(K,g,\ell)$, the reduction hypothesis ($\mathrm{RT}_{\mathrm{red}}$) is not imposed on elements in $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$. Our main result in this paper is the following theorem.

THEOREM 1.2. *The set $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ is empty for any prime number ℓ large enough.*

Since $\mathcal{A}(K,g,\ell)_{\mathrm{ab}} \subset \mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$, we have the following corollary.

COROLLARY 1.3. *The set $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ is empty for any prime number ℓ large enough.*

Keys to the proof of Theorem 1.2 are to construct a compatible system of Galois representations which has a strong condition, Faltings' trick in his proof of the Shafarevich Conjecture and Raynaud's criterion of semistable reduction. We hope that this study will be a first step to solve Conjecture 1.1 for abelian varieties with complex multiplication. If an abelian variety A over K has complex multiplication over K (in the sense of [20, Section 4]), then it is well-known that $\rho_{A,\ell}$ is abelian (cf. [*loc. cit.*, Section 4, Corollary 2]). Thus we obtain

COROLLARY 1.4. *The set of K-isomorphism classes of abelian varieties in $\mathcal{A}(K,g,\ell)$ which have complex multiplication over K is empty for any prime number ℓ large enough.*

We want to replace "an abelian image" in the statement of ($\mathrm{RT}_{\mathrm{ab}}$) with "a potential abelian image". If Theorem 1.2 with this replacement is proved, then Conjecture 1.1 holds for CM abelian varieties, that is, if we denote by $\mathcal{A}(K,g,\ell)_{\mathrm{CM}}$ the set of K-isomorphism classes of abelian varieties in $\mathcal{A}(K,g,\ell)$ which have complex multiplication over $\bar{K}$, then the set

$$\mathcal{A}(K,g)_{\mathrm{CM}} := \{([A],\ell); [A] \in \mathcal{A}(K,g,\ell)_{\mathrm{CM}},\ \ell : \text{prime number}\}$$

is finite.

The paper proceeds as follows. Section 2 is devoted to a study of compatible systems. In Section 3, we recall some facts about Conjecture 1.1. Finally we prove our main theorem in Section 4.

2. Compatible systems

In this section, we use the notation given in Introduction. It is important for the proof of Theorem 1.2 to find conditions for the compatible systems to be of a simple form.

2.1. **Basic notions.** Let E be a finite extension of $\mathbb{Q}$. For a finite place λ of E, we denote by ℓ_λ the prime number below λ, E_λ the completion of E at λ and $\mathbb{F}_\lambda$ the residue field of λ. We denote by K_v the completion of K at a finite place v of K. Let S be a finite set of finite places of K and T a finite set of finite places of E. Put $S_\ell = S \cup \{\text{places of } K \text{ above } \ell\}$. A representation $\rho\colon G_K \to GL_n(E_\lambda)$ is said to be *E-rational (resp. E-integral) with ramification set S* if ρ is unramified outside S_{ℓ_λ} and the characteristic polynomial $\det(XI_n - \rho(\mathrm{Fr}_v))$ of Fr_v has coefficients in E (resp. the integer ring of E) for each finite place $v \notin S_{\ell_\lambda}$ of K, where Fr_v is an arithmetic Frobenius of v and I_n is the identity matrix of size n.

Now we give definitions of compatible systems of λ-adic (resp. mod λ) representations, which mainly follow those in [10] and [11].

Definition 2.1. (1) An *E-rational (resp. E-integral) strictly compatible system $(\rho_\lambda)_\lambda$ of n-dimensional λ-adic representations of G_K with defect set T and ramification set S* is a family of continuous representations $\rho_\lambda\colon G_K \to GL_n(E_\lambda)$ for finite places λ of E not in T such that

- (i) ρ_λ is unramified outside S_{ℓ_λ};
- (ii) for any finite place $v \notin S$ of K, there exists a monic polynomial $f_v(X) \in E[X]$ (resp. $f_v(X) \in E[X]$ with coefficients in the integer ring of E) such that for all finite places $\lambda \notin T$ of E coprime to the residue characteristic of v, the characteristic polynomial $\det(XI_n - \rho_\lambda(\mathrm{Fr}_v))$ of Fr_v is equal to $f_v(X)$.

(2) An *E-rational (resp. E-integral) strictly compatible system $(\bar\rho_\lambda)_\lambda$ of n-dimensional mod λ representations of G_K with defect set T and ramification set S* is a family of continuous representations $\bar\rho_\lambda\colon G_K \to GL_n(\mathbb{F}_\lambda)$ for finite places λ of E not in T such that

- (i) $\bar\rho_\lambda$ is unramified outside S_{ℓ_λ};
- (ii) for any finite place $v \notin S$ of K, there exists a monic polynomial $f_v(X) \in E[X]$ (resp. $f_v(X) \in E[X]$ with coefficients in the integer ring of E) such that for all finite places $\lambda \notin T$ of E coprime to the residue characteristic of v, $f_v(X)$ is integral at λ and the characteristic polynomial $\det(XI_n - \bar\rho_\lambda(\mathrm{Fr}_v))$ of Fr_v is equal to the reduction of $f_v(X)$ mod λ.

We will often suppress the sets S and T from the notations.

Example 2.2. Let X be a proper smooth variety over K. Let $V_\ell := H^r_{\text{ét}}(X_{\bar K}, \mathbb{Q}_\ell)^\vee$ be the dual of the ℓ-adic étale cohomology group $H^r_{\text{ét}}(X_{\bar K}, \mathbb{Q}_\ell)$ of X. Then the system $(V_\ell)_\ell$ is a $\mathbb{Q}$-rational strictly compatible system whose defect set is empty and whose ramification set is the set of finite places of K where X has bad reduction. This fact follows from the Weil Conjecture which is proved by Deligne (cf. [5], [6]).

It is conjectured that every E-rational strictly compatible system arises motivically.

CONJECTURE 2.3 ([10, Conjecture 1]). *Any E-rational strictly compatible system of λ-adic (resp. mod λ) representations arises motivically.*

In fact, this conjecture is true if the representations are abelian.

THEOREM 2.4 ([11, Theorem 2, Corollary 1]). *An E-rational strictly compatible system of abelian semisimple λ-adic (resp. mod λ) representations of G_K arises from Hecke characters.*

For informations of Hecke characters, see the next subsection.

For a finite place v of K, we denote by G_v the decomposition group at v (where we fix an embedding $\bar{K} \hookrightarrow \bar{K}_v$) and by I_v the inertia subgroup of G_v. An *inertial level $\mathfrak{L}$ of K* is a collection $(\mathfrak{L}_v)_v$ of open normal subgroups $\mathfrak{L}_v$ of I_v for each finite place v of K such that $\mathfrak{L}_v = I_v$ for almost all v. We say that a λ-adic representation of G_K is *geometric* (cf. [8]) if

(i) it is unramified outside a finite set of places of K;
(ii) its restriction to G_v for any finite place v of K is potentially semistable.

The *inertial level $\mathfrak{L}$ of a geometric λ-adic representation ρ_λ of G_K* is the collection $(\mathfrak{L}_v(\rho_\lambda))_v$ of open normal subgroups $\mathfrak{L}_v(\rho_\lambda)$ of I_v for each finite place v of K, where $\mathfrak{L}_v(\rho_\lambda)$ is the largest open subgroup of I_v such that the restriction of ρ_λ to $\mathfrak{L}_v(\rho_\lambda)$ is semistable. By the definition of geometric Galois representations, we have $\mathfrak{L}_v(\rho_\lambda) = I_v$ for almost all v. A compatible system $(\rho_\lambda)_\lambda$ of geometric λ-adic representations of G_K has *bounded inertial level* if there exists an inertial level $\mathfrak{L} = (\mathfrak{L}_v)_v$ such that $\mathfrak{L}_v \subset \mathfrak{L}_v(\rho_\lambda)$ for all λ and v. Let $w_1, w_2, \dots, w_n$ be integers. A λ-adic representation ρ_λ is *E-rational* (*resp. E-integral*) *with Frobenius weights $w_1, w_2, \dots, w_n$ outside S* if ρ_λ is E-rational (resp. E-integral) with ramification set S and, for all finite places $v \notin S_{\ell_\lambda}$ of K, the complex roots of the characteristic polynomial $\det(XI_n - \rho_\lambda(\mathrm{Fr}_v))$ of Fr_v, for a chosen embedding of E into the field of complex numbers $\mathbb{C}$, have complex absolute values $q_v^{w_1/2}, q_v^{w_2/2}, \dots, q_v^{w_n/2}$, where q_v is the cardinality of the residue field of v. An E-rational strictly compatible system $(\rho_\lambda)_\lambda$ is said to be an *E-rational* (*resp. E-integral*) *strictly compatible system with Frobenius weights* $w_1, w_2, \dots, w_n$ if each ρ_λ is E-rational (resp. E-integral) with Frobenius weights $w_1, w_2, \dots, w_n$ outside the ramification set of $(\rho_\lambda)_\lambda$. We call $w_1, w_2, \dots, w_n$ the Frobenius weights of ρ_λ (resp. $(\rho_\lambda)_\lambda$), and ρ_λ (resp. $(\rho_\lambda)_\lambda$) is said to be *pure* if $w_1 = w_2 = \cdots = w_n$. A compatible system $(\rho_\lambda)_\lambda$ of geometric λ-adic representations of G_K has *bounded Hodge-Tate weights* if there exist integers a and b with $a \leq b$ such that, for any λ and any finite place v of K above ℓ_λ, all the Hodge-Tate weights of $\rho_\lambda|_{G_v}$ viewed as a $\mathbb{Q}_{\ell_\lambda}$-representation are in $[a, b]$. Finally, a compatible system $(\bar{\rho}_\lambda)_\lambda$ of mod λ representations of G_K is of *bounded Artin conductor* if there exists a non-zero ideal $\mathfrak{n}$ of $\mathcal{O}_K$ such that, for any λ, the Artin conductor outside ℓ_λ of $\bar{\rho}_\lambda$ divides $\mathfrak{n}$.

PROPOSITION 2.5. (1) *An E-rational strictly compatible system $(\rho_\lambda)_\lambda$ of abelian semisimple λ-adic representations of G_K has bounded inertial level and bounded Hodge-Tate weights.*
(2) *An E-rational strictly compatible system $(\bar{\rho}_\lambda)_\lambda$ of abelian semisimple mod λ representations of G_K is of bounded Artin conductor.*

PROOF. By Theorem 2.4, such compatible systems arise from Hecke characters. Hence the proposition follows from standard properties of a representation arising from Hecke characters (see Proposition 2.7 below). □

2.2. **Compatible systems arising from Hecke characters.** In this subsection, we recall the construction of Galois representations arising from Hecke characters and their standard properties (cf. [10, Section 4.1], [17, Chapter I], [19, Chapter II] and [22]). We index as usual the real (resp. complex) places of K by an embedding σ of K into $\mathbb{R}$ (resp. pairs of elements $\{\sigma, c\sigma\}$ where σ is an embedding of K into $\mathbb{C}$ with $\sigma(K) \not\subset \mathbb{R}$ and c is complex conjugation). We denote by I_K the idele group of K and by $(K_\infty^\times)^0 \subset I_K$ the connected component of $K_\infty^\times$ which contains the identity, where $K_\infty^\times$ is the product of the unit groups of the completions of K at the infinite places. For any $z \in \mathbb{C}$, denote by $\bar{z}$ the complex conjugation of z.

DEFINITION 2.6. A *Hecke character* is a continuous homomorphism $\psi\colon I_K \to \mathbb{C}^\times$ such that

(i) $\psi(K^\times) = \{1\}$;
(ii) there exist integers $n_\sigma, n_{c\sigma}$ which satisfy

$$\psi|_{(K_\infty^\times)^0}(x) = \prod_{\sigma \text{ real}} x_\sigma^{n_\sigma} \prod_{\sigma \text{ complex}} x_\sigma^{n_\sigma} \bar{x}_\sigma^{n_{c\sigma}}$$

with x_σ the components of x.

We say that the family of integers $(n_\sigma)_\sigma$ is the *infinity type of* ψ. The *conductor of* ψ is the largest ideal $\mathfrak{n}$ such that elements of the finite ideles $I^{(\infty)}$ of K congruent to 1 mod $\mathfrak{n}$ are in the kernel of ψ.

Now we construct Galois representations arising from a Hecke character ψ. We want to use class field theory: $G_K^{\mathrm{ab}} \simeq I_K/\overline{K^\times (K_\infty^\times)^0}$, where G_K^{ab} is the Galois group of the maximal abelian extension of K over K and $\overline{K^\times (K_\infty^\times)^0}$ is the topological closure of $K^\times (K_\infty^\times)^0$. Let ℓ be a prime number. Let $\psi_0\colon I_K \to \mathbb{C}^\times$ be the homomorphism defined by

$$\psi_0(x) = \psi(x) \prod_{\sigma \text{ real}} x_\sigma^{-n_\sigma} \prod_{\sigma \text{ complex}} x_\sigma^{-n_\sigma} \bar{x}_\sigma^{-n_{c\sigma}},$$

then its kernel is open and takes values in a sufficiently large finite extension E (in $\mathbb{C}$) of $\mathbb{Q}$, and thus we may regard ψ_0 as a homomorphism $I_K \to E^\times$. By definition, ψ_0 factors through the quotient $I_K/(K_\infty^\times)^0$, however, ψ_0 is not trivial on $K^\times$. We modify ψ_0 by changing the image of the ℓ-part $K_\ell^\times := (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \simeq \prod_{v|\ell} K_v^\times$ of I_K. Suppose that E contains the Galois closure of K over $\mathbb{Q}$ and take any finite place λ of E above ℓ. Let $\eta\colon K^\times \to E_\lambda^\times$ be the homomorphism defined by $\eta(x) = \prod_\sigma \sigma(x)^{n_\sigma}$, where σ runs through all embeddings $K \hookrightarrow \mathbb{C}$, and extend η to $\eta_\ell\colon (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \to E_\lambda^\times$. Then we have the continuous homomorphism $\psi_\lambda := \psi_0 \cdot (\eta_\ell \circ \alpha_\ell)$, where $\alpha_\ell\colon I_K \to (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$ is the projection. Using the isomorphism of class field theory $G_K^{\mathrm{ab}} \simeq I_K/\overline{K^\times(K_\infty^\times)^0}$, we obtain a continuous character $\psi_\lambda\colon G_K \to E_\lambda^\times$. Since ψ_λ is continuous, we know that ψ_λ has values in the group of units of E_λ and thus we obtain a mod λ representation $\bar{\psi}_\lambda\colon G_K \to \mathbb{F}_\lambda^\times$. Denote by π_v a uniformizer of K_v for any finite place v. By construction, we see the following standard properties for Galois representations arising from Hecke characters.

PROPOSITION 2.7. *Let the notation be as above.*
(1) *The character ψ_λ is unramified away from $\ell\mathfrak{n}$ where $\mathfrak{n}$ is the conductor of ψ. Moreover, ψ_λ is locally algebraic in the sense of [19, Chapter III] (See also [17, Chapter I, Section 5]), and hence it is geometric (cf. [8, Section 6, Proposition]).*
(2) *For any finite place v away from $\ell\mathfrak{n}$, we have equalities*

$$\psi_\lambda(\mathrm{Fr}_v) = \psi_\lambda(\pi_v) = \psi(\pi_v) = \psi_0(\pi_v) \in E.$$

In particular, $\psi_\lambda(\mathrm{Fr}_v)$ is independent of the choice of λ and has values in E.
(3) *The system $(\psi_\lambda)_\lambda$ forms an E-rational strictly compatible system of 1-dimensional λ-adic representations of G_K with bounded inertial level and bounded Hodge-Tate weights.*
(4) *The system $(\bar{\psi}_\lambda)_\lambda$ forms an E-rational strictly compatible system of 1-dimensional mod λ representations of G_K of bounded Artin conductor.*

PROOF. We only give a very rough sketch: To see locally algebraicity of (1), for example, use [17, Proposition (1.5.4)]. Boundednesses of the inertial level and the Artin conductor in (3) and (4) follow from the existence of the conductor of Hecke characters. The boundedness of Hodge-Tate weights in (3) can be seen from the infinite type of Hecke characters, or, for example, we can also check the boundedness by combining locally algebraicity of (1) and the arguments given in the last part (around p. 482) of the proof of [21, Theorem]. All the other statements follow from direct calculations. □

2.3. **Structures of certain compatible systems.** Choose an algebraic closure $\bar{\mathbb{F}}_\lambda$ of $\mathbb{F}_\lambda$. Put $\chi_\lambda\colon G_K \xrightarrow{\chi_{\ell_\lambda}} \mathbb{Z}_{\ell_\lambda}^\times \hookrightarrow E_\lambda^\times$ and $\bar{\chi}_\lambda\colon G_K \xrightarrow{\bar{\chi}_{\ell_\lambda}} \mathbb{F}_{\ell_\lambda}^\times \hookrightarrow \mathbb{F}_\lambda^\times$, where χ_{ℓ_λ} and $\bar{\chi}_{\ell_\lambda}$ are the ℓ_λ-adic cyclotomic character and the mod ℓ_λ cyclotomic character, respectively. For a representation $\bar{\rho}_\lambda\colon G_K \to GL_n(\mathbb{F}_\lambda)$ with abelian semisimplification, the representation $(\bar{\rho}_\lambda)^{\mathrm{ss}} \otimes \bar{\mathbb{F}}_\lambda$ is conjugate to the direct sum of n characters, where the subscript "ss" means the semisimplification, and we call these n characters the *characters associated with* $\bar{\rho}_\lambda$

(remark that Schur's lemma implies that any irreducible abelian $\bar{\mathbb{F}}_\lambda$-representation is of dimension one). For a λ-adic representation ρ_λ, we denote by $\bar{\rho}_\lambda$ a residual representation of ρ_λ (for a chosen lattice). Note that the isomorphism class of $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ is independent of the choice of a lattice by the Brauer-Nesbitt theorem.

THEOREM 2.8. *Let $(\rho_\lambda)_\lambda$ be an E-rational strictly compatible system of n-dimensional geometric semisimple λ-adic representations of G_K. Suppose that there exists an infinite set Λ of finite places of E which satisfies the following conditions.*

(1) *For any $\lambda \in \Lambda$, there exists a place v_λ of K above ℓ_λ such that*
  (a) *ρ_λ is semistable at v_λ;*
  (b) *there exist integers $w_1 \leq w_2$ such that the Hodge-Tate weights of $\rho_\lambda|_{G_{v_\lambda}}$ are in $[w_1, w_2]$ for any pair (λ, v_λ).*
(2) *For any $\lambda \in \Lambda$, $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ is abelian and any character associated with $\bar{\rho}_\lambda$ has the form $\varepsilon\bar{\chi}_\lambda^a$, where a is an integer and $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\lambda^\times$ is a character unramified at all places of K above ℓ_λ.*
(3) *The Artin conductor of $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ is bounded independently of the choice of $\lambda \in \Lambda$. That is, there exists a non-zero ideal $\mathfrak{n}$ of $\mathcal{O}_K$ such that, for any $\lambda \in \Lambda$, the Artin conductor outside ℓ_λ of $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ divides $\mathfrak{n}$.*

Then there exist integers $m_1, m_2, \ldots, m_n$ and a finite extension L of K such that, for any λ, the representation ρ_λ is isomorphic to $\chi_\lambda^{m_1} \oplus \chi_\lambda^{m_2} \oplus \cdots \oplus \chi_\lambda^{m_n}$ on G_L.

REMARK 2.9. If Conjecture 2.3 holds, then we can remove the conditions (1) and (3) of Theorem 2.8 since these conditions are automatically satisfied.

LEMMA 2.10. *Let $\mathbb{F}$ be a field of characteristic $\ell > 0$. Let ρ and ρ' be n-dimensional semisimple $\mathbb{F}$-representations of a group G. Assume that $\ell > n$. If $\mathrm{Tr}(\rho(g)) = \mathrm{Tr}(\rho'(g))$ for any $g \in G$, then ρ is isomorphic to ρ'.*

PROOF. Let V and V' be underlying $\mathbb{F}$-vector spaces of ρ and ρ', respectively. Let $\mathfrak{S}$ be the category of isomorphism classes of simple $\mathbb{F}[G]$-modules. Then there exist a finite set H of objects of $\mathfrak{S}$ and integers $n_h, n'_h \geq 0$ for each $h \in H$ such that

$$V \simeq \bigoplus_{h \in H} W_h^{\oplus n_h}, \quad V' \simeq \bigoplus_{h \in H} W_h^{\oplus n'_h}$$

as $\mathbb{F}[G]$-modules. Here, W_h is a (chosen) representative of h. By a method similar to the proof of [1, Section 12, n° 1, Proposition 3], we obtain $(n_h - n'_h)\dim_{\mathbb{F}}(W_h) = 0$ in $\mathbb{F}$. Since $\ell > n$, we know that $\dim_{\mathbb{F}}(W_h) \in \mathbb{F}^\times$ and thus $n_h - n'_h \equiv 0 \bmod \ell$. Since $-n \leq n_h - n'_h \leq n$, by using the assumption $\ell > n$ again, we obtain $n_h = n'_h$. □

Proof of Theorem 2.8. By taking a positive integer m large enough and twisting χ_λ^m to ρ_λ for all λ, we may assume $w_1 \geq 0$. Furthermore, by replacing Λ with its infinite subset, we may suppose that ℓ_λ does not divide the discriminant of K and $\ell_\lambda > [E : \mathbb{Q}] \cdot n$ for any $\lambda \in \Lambda$. By the condition (3), there exists a non-zero ideal $\mathfrak{n}$ of $\mathcal{O}_K$ such that, for any $\lambda \in \Lambda$, the Artin conductor outside ℓ_λ of $(\bar{\rho}_\lambda)^{\rm ss}$ divides $\mathfrak{n}$. If we denote by ψ a character associated with $(\bar{\rho}_\lambda)^{\rm ss}$ for $\lambda \in \Lambda$ and decompose $\psi = \varepsilon\bar{\chi}_\lambda^a$, where ε is as in the condition (2), then the Artin conductor outside ℓ_λ of ε also divides $\mathfrak{n}$. Hence, replacing the field K with the strict ray class field of K associated with $\mathfrak{n}$, we may replace the condition (2) with the following condition $(2)'$:

$(2)'$ For any $\lambda \in \Lambda$, $(\bar{\rho}_\lambda)^{\rm ss}$ is abelian and any character associated with $\bar{\rho}_\lambda$ has the form $\bar{\chi}_\lambda^a$.

Now take any $\lambda \in \Lambda$. Let $\bar{\chi}_\lambda^{a_{\lambda,1}}, \bar{\chi}_\lambda^{a_{\lambda,2}}, \dots, \bar{\chi}_\lambda^{a_{\lambda,n}}$ be all the characters associated with $\bar{\rho}_\lambda$. By Lemma 2.10 together with the condition $(2)'$ and $\ell_\lambda > [E:\mathbb{Q}]\cdot n$, we see that the representation $(\bar{\rho}_\lambda)^{\rm ss}$ is conjugate to the direct sum of n characters (over $\mathbb{F}_\lambda$) of the form $\bar{\chi}_\lambda^a$ which has values in $\mathbb{F}_{\ell_\lambda}^\times$. Hence if we regard the $\mathbb{F}_\lambda$-representation $\bar{\rho}_\lambda$ as an $\mathbb{F}_{\ell_\lambda}$-representation, its semisimplification is of a diagonal form whose diagonal components are the copies of $\bar{\chi}_{\ell_\lambda}^{a_{\lambda,1}}, \bar{\chi}_{\ell_\lambda}^{a_{\lambda,2}}, \dots, \bar{\chi}_{\ell_\lambda}^{a_{\lambda,n}}$ (here we note that $\ell_\lambda > [\mathbb{F}_\lambda : \mathbb{F}_{\ell_\lambda}]\cdot n$). Furthermore, it is a direct summand of the semisimplification of a residual representation of ρ_λ viewed as a $\mathbb{Q}_{\ell_\lambda}$-representation. Therefore, by the condition (1) and applying Caruso's result on an upper bound for tame inertia weights (cf. [4] or [16, Theorem 2.5 and Remark 2.6]) to $\bar{\rho}_\lambda|_{G_{v_\lambda}}$ (viewed as an $\mathbb{F}_{\ell_\lambda}$-representation), there exists an integer $0 \le b_{\lambda,i} \le w_2$ such that

$$(\sharp) \quad b_{\lambda,i} \equiv a_{\lambda,i} \bmod \ell_\lambda - 1$$

for any i (recall that ℓ_λ does not divide the discriminant of K). Now we claim that the set $\{b_{\lambda,1}, b_{\lambda,2}, \dots, b_{\lambda,n}\}$ is independent of the choice of $\lambda \in \Lambda$ large enough. Denote by S the ramification set of $(\rho_\lambda)_\lambda$. Take a $v_0 \notin S$ and decompose $\det(XI_n - \rho_\lambda(\mathrm{Fr}_{v_0})) = \prod_{j=1}^n (X - \alpha_{v_0,j})$. By the conditions $(2)'$ and $(\sharp)$, we have the congruence $\prod_{j=1}^n (X - \alpha_{v_0,j}) \equiv \prod_{j=1}^n (X - q_{v_0}^{b_{\lambda,j}})$ in $\bar{\mathbb{F}}_\lambda[X]$. If ℓ_λ is large enough (note that Λ is an infinite set), then we obtain that this congruence is in fact an equality in $E[X]$: $\prod_{j=1}^n (X - \alpha_{v_0,j}) = \prod_{j=1}^n (X - q_{v_0}^{b_{\lambda,j}})$. Therefore, the set $\{b_{\lambda,1}, b_{\lambda,2}, \dots, b_{\lambda,n}\}$ is independent of the choice of $\lambda \in \Lambda$ with ℓ_λ large enough. This proves the claim. We denote $\{b_{\lambda,1}, b_{\lambda,2}, \dots, b_{\lambda,n}\}$ by $\{m_1, m_2, \dots, m_n\}$ for such $\lambda' s$. Since $(\rho_\lambda)_\lambda$ is a compatible system, we obtain the equation $\det(XI_n - \rho_\lambda(\mathrm{Fr}_v)) = \prod_{j=1}^n (X - q_v^{m_j})$ for any λ and any $v \notin S_{\ell_\lambda}$. Therefore, the representation ρ_λ is isomorphic to $\chi_\lambda^{m_1} \oplus \chi_\lambda^{m_2} \oplus \cdots \oplus \chi_\lambda^{m_n}$. This finishes the proof. ☐

Corollary 2.11. *Let $(\bar{\rho}_\lambda)_\lambda$ be an E-rational strictly compatible system of abelian semisimple mod λ representations of G_K. Suppose that, for infinitely many finite places λ of E, any character associated with $\bar{\rho}_\lambda$ has the form $\varepsilon\bar{\chi}_\lambda^a$, where $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\lambda^\times$ is a character unramified at all places of K above ℓ_λ. Then there exist a finite extension L of K and integers $m_1, m_2, \dots, m_n$ such that, for any λ, the representation $\bar{\rho}_\lambda$ is isomorphic to $\bar{\chi}_\lambda^{m_1} \oplus \bar{\chi}_\lambda^{m_2} \oplus \cdots \oplus \bar{\chi}_\lambda^{m_n}$ on G_L.*

PROOF. By Theorem 2.4, we know that there exist a finite extension E' of E and an E'-rational abelian semisimple compatible system $(\rho_{\lambda'})_{\lambda'}$ of λ'-adic representations of G_K which arises from Hecke characters such that $(\rho_{\lambda'})_{\lambda'}$ is a lift of $(\bar{\rho}_\lambda)_\lambda$, that is, a residual representation $\bar{\rho}_{\lambda'}$ of $\rho_{\lambda'}$ is isomorphic to $\bar{\rho}_\lambda \otimes \mathbb{F}_{\lambda'}$ for any λ and any finite place λ' of E' above λ. By Proposition 2.7, we see that $(\rho_{\lambda'})_{\lambda'}$ satisfies all the assumptions (1), (2) and (3) in Theorem 2.8. Consequently we obtain the desired result. □

COROLLARY 2.12. *Let $(\rho_\lambda)_\lambda$ be an E-rational strictly compatible system of n-dimensional semisimple λ-adic representations of G_K. Suppose that*

(i) $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ *is abelian for almost all* λ*;*
(ii) *for infinitely many λ, any character associated with $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ has the form $\varepsilon\bar{\chi}_\lambda^a$, where $\varepsilon\colon G_K \to \mathbb{F}_\lambda^\times$ is a character unramified at all places of K above ℓ_λ.*

Then there exist integers $m_1, m_2, \dots, m_n$ and a finite extension L of K such that, for any λ, the representation ρ_λ is isomorphic to $\chi_\lambda^{m_1} \oplus \chi_\lambda^{m_2} \oplus \cdots \oplus \chi_\lambda^{m_n}$ on G_L.

PROOF. For any finite place v of K not in the ramification set of $(\rho_\lambda)_\lambda$, let $f_v(X)$ be as in Definition 2.1 (1). Applying Corollary 2.11 to the compatible system $((\bar{\rho}_\lambda)^{\mathrm{ss}})_\lambda$, we see that there exist a finite extension L of K and integers $m_1, m_2, \dots, m_n$ such that, for any λ, the representation $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ is isomorphic to $\bar{\chi}_\lambda^{m_1} \oplus \bar{\chi}_\lambda^{m_2} \oplus \cdots \oplus \bar{\chi}_\lambda^{m_n}$ on G_L. Thus we have $f_v(X) = \prod_{j=1}^n (X - q_v^{m_j})$ in $\mathbb{F}_\lambda[X]$ for any λ coprime to v. Taking λ with ℓ_λ large enough, we see that $f_v(X) = \prod_{j=1}^n (X - q_v^{m_j})$ in $E[X]$ and the result follows. □

Let λ and λ' be finite places of E of different residual characteristics. Let ρ_λ be an E-rational n-dimensional semisimple λ-adic representation of G_K with ramification set S. Suppose that there exists a semisimple λ'-adic representation $\rho_{\lambda'}$ of G_K such that

$$\det(XI_n - \rho_\lambda(\mathrm{Fr}_v)) = \det(XI_n - \rho_{\lambda'}(\mathrm{Fr}_v))$$

for any $v \notin S_{\ell_\lambda} \cup S_{\ell_{\lambda'}}$. In the spirit of Fontaine-Mazur's "Main Conjecture", we hope that $\rho_{\lambda'}$ is crystalline for any finite place v' of K above $\ell_{\lambda'}$ when the residual characteristic of λ' is prime to that of any place in S. However, to prove this hope seems not to be easy. If we consider representations which are pure, we can improve the statement (1) of Theorem 2.8 as below. (If the hope is true, it is not difficult to prove the proposition below without the assumption of pureness by a method similar to the proof of Theorem 2.8.)

PROPOSITION 2.13. *Let $(\rho_\lambda)_\lambda$ be an E-rational strictly compatible system of n-dimensional geometric semisimple λ-adic representations of G_K. Suppose that $(\rho_\lambda)_\lambda$ is pure. Suppose that there exists an infinite set Λ of finite places of K which satisfies the following conditions.*

(1) *For any $\lambda \in \Lambda$, there exists a place v_λ of K above ℓ_λ such that*
 (a) *there exists a constant $C > 0$ such that $[I_{v_\lambda} : \mathfrak{L}_{v_\lambda}(\rho_\lambda)] < C$ for any pair (λ, v_λ). Here $\mathfrak{L}_{v_\lambda}(\rho_\lambda)$ is the inertial level of ρ_λ at v_λ;*

(b) *there exist integers* $w_1 \leq w_2$ *such that the Hodge-Tate weights of* $\rho_\lambda|_{G_{v_\lambda}}$ *are in* $[w_1, w_2]$ *for any pair* (λ, v_λ).

(2) *For any* $\lambda \in \Lambda$, $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ *is abelian and any character associated with* $\bar{\rho}_\lambda$ *has the form* $\varepsilon\bar{\chi}_\lambda^a$, *where* $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\lambda^\times$ *is a character unramified at all places of* K *above* ℓ_λ.

(3) *The Artin conductor of* $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ *is bounded independently of the choice of* $\lambda \in \Lambda$ *in the sense of* (3) *of Theorem* 2.8.

Then there exist an integer m *and a finite extension* L *of* K *such that, for any* λ, *the representation* ρ_λ *is isomorphic to* $(\chi_\lambda^m)^{\oplus n}$ *on* G_L.

PROOF. Most parts of the first paragraph of this proof will proceed by a method similar to the proof of Theorem 2.8 and hence we will often omit precise arguments. First we may assume that, for any $\lambda \in \Lambda$,

(2)′ any character associated with $\bar{\rho}_\lambda$ has the form $\bar{\chi}_\lambda^a$,

and furthermore, there exists a positive integer r such that $\rho_\lambda|_{G_{v_\lambda}}$ has Hodge-Tate weights in $[0, r]$ for any pair (λ, v_λ) as in the condition (1). Suppose λ is a finite place in Λ. Let $\bar{\chi}_\lambda^{a_{\lambda,1}}, \bar{\chi}_\lambda^{a_{\lambda,2}}, \dots, \bar{\chi}_\lambda^{a_{\lambda,n}}$ be all the characters associated with $\bar{\rho}_\lambda$. Taking a finite place v_λ as in the condition (1), there exists a finite extension $L_{w(\lambda)}$ of K_{v_λ} such that $\rho_\lambda|_{G_{L_{w(\lambda)}}}$ is semistable and $[L_{w(\lambda)} : K_{v_\lambda}] \leq C$. If we denote by $e_{w(\lambda)}$ the absolute ramification index of $L_{w(\lambda)}$, then we have $e_{w(\lambda)} \leq C[K : \mathbb{Q}]$, and Caruso's result on an upper bound for tame inertia weights of $\bar{\rho}_\lambda|_{G_{L_{w(\lambda)}}}$ (viewed as an $\mathbb{F}_{\ell_\lambda}$-representation) implies that there exists an integer $0 \leq b'_{\lambda,i} \leq e_{w(\lambda)} r$ which satisfies $b'_{\lambda,i} \equiv e_{w(\lambda)} a_{\lambda,i} \bmod \ell_\lambda - 1$. Putting $e = lcm_{\lambda\in\Lambda}(e_{w(\lambda)})$ and $b_{\lambda,i} = b'_{\lambda,i} e/e_{w(\lambda)}$, we have $e \leq (C[K : \mathbb{Q}])!$, $b_{\lambda,i} \in [0, er]$ and $b_{\lambda,i} \equiv e a_{\lambda,i} \bmod \ell_\lambda - 1$. Note that e is independent of the choice of $\lambda \in \Lambda$. Take any $v \notin S_{\ell_\lambda}$ and decompose $\det(XI_n - \rho_\lambda(\mathrm{Fr}_v)) = \prod_{j=1}^n (X - \alpha_{v,j})$. Then, by an argument similar to that in the proof of Theorem 2.8, we can show that $\prod_{j=1}^n (X - \alpha_{v,j}^e) = \prod_{j=1}^n (X - q_v^{b_{\lambda,j}})$ if we take $\lambda \in \Lambda$ with ℓ_λ large enough. Since $(\rho_\lambda)_\lambda$ is pure, we know that $b_{\lambda,1} = \cdots = b_{\lambda,n}$ and furthermore they are independent of the choice of $\lambda \in \Lambda$. Putting $b := b_{\lambda,1} = \cdots = b_{\lambda,n}$, we have

$$\prod_{j=1}^n (X - \alpha_{v,j}^e) = \prod_{j=1}^n (X - q_v^b). \qquad (*)$$

Fix $\lambda \in \Lambda$ and denote it by λ_0. By taking a finite extension K' of K large enough, we can define a continuous character $\chi_{\lambda_0}^{1/e}\colon G_{K'} \to E_{\lambda_0}^\times$ which has values in the integer ring of E_{λ_0} and $(\chi_{\lambda_0}^{1/e})^e = \chi_{\lambda_0}$. In fact, this can be checked as follows: Let $\mathfrak{m}_{\lambda_0}$ be the maximal ideal of the integer ring of E_{λ_0}. Let e_{λ_0} be the absolute ramification index of E_{λ_0}. Fix an integer k which satisfies $k > e_{\lambda_0}/(\ell_{\lambda_0} - 1)$, and take a finite extension K' of K such that $\chi_{\lambda_0}(G_{K'}) \subset 1 + e\mathfrak{m}_{\lambda_0}^k$. Then we obtain the desired character $\chi_{\lambda_0}^{1/e}$ by the composite

$$G_{K'} \overset{\chi_{\lambda_0}}{\to} 1 + e\mathfrak{m}_{\lambda_0}^k \overset{\log}{\to} e\mathfrak{m}_{\lambda_0}^k \overset{1/e}{\to} \mathfrak{m}_{\lambda_0}^k \overset{\exp}{\to} 1 + \mathfrak{m}_{\lambda_0}^k \subset E_{\lambda_0}^\times.$$

In the argument below, we use the method of the proof of [12, Proposition 1.2]. Let λ_0 and K' be as above and replace K with this K'. The equation $(*)$ implies that, for any $v \notin S_{\ell_{\lambda_0}}$, all the roots of $\det(XI_n - \rho'_{\lambda_0}(\mathrm{Fr}_v))$ are e-th roots of unity, where ρ'_{λ_0} is the twist of ρ_{λ_0} by $(\chi_{\lambda_0}^{1/e})^{-b}$. In particular, there are only finitely many possibilities for the characteristic polynomial of Fr_v. Now the function which takes $g \in G_K$ to $\det(XI_n - \rho'_{\lambda_0}(g)) \in E_{\lambda_0}[X]$ is continuous (here, we equip $E_{\lambda_0}[X] \simeq \bigoplus_{i \geq 0} E_{\lambda_0} X^i$ with the direct product topology of the λ_0-adic topology on $E_{\lambda_0} X^i$), and takes only finitely many values by Chebotarev's density theorem. It follows that the set $\{g \in G_K; \det(XI_n - \rho'_{\lambda_0}(g)) = (X-1)^n\}$ is an open subset of G_K, which contains the identity map of $\bar{K}$. Hence there exists a finite extension L of K such that $G_L \subset \{g \in G_K; \det(XI_n - \rho'_{\lambda_0}(g)) = (X-1)^n\}$. Then we see that ρ_{λ_0} is isomorphic to $((\chi_{\lambda_0}^{1/e})^b)^{\oplus n}$ on G_L. Since ρ_{λ_0} is geometric, ρ_{λ_0} is Hodge-Tate at any place of L above ℓ_{λ_0} and thus we know that $b/e =: m$ is an integer. This finishes the proof. □

3. Rasmussen-Tamagawa Conjecture

We continue to use the same notation as in the previous section. Let $g \geq 0$ be an integer. For any abelian variety A over K, denote by $A[\ell]$ the group of $\bar{K}$-valued ℓ-torsion points of A and by $K(A[\ell])$ the field generated by K and the coordinates of $A[\ell]$.

Definition 3.1. We denote by $\mathcal{A}(K, g, \ell)$ the set of K-isomorphism classes of g-dimensional abelian varieties A over K which satisfy the following conditions.

(RT_ℓ) $K(A[\ell])$ is an ℓ-extension of $K(\mu_\ell)$.

($\mathrm{RT}_{\mathrm{red}}$) The abelian variety A has good reduction away from ℓ over K.

By ($\mathrm{RT}_{\mathrm{red}}$), the set $\mathcal{A}(K, g, \ell)$ is a finite set (cf. [7, Theorem 5] and [23, 1. Theorem]). Rasmussen and Tamagawa conjectured in [18] that for any ℓ large enough, this set is in fact empty (see Conjecture 1.1 in Introduction). The following results on the Rasmussen-Tamagawa Conjecture are known:

(i) ([18, Theorem 2]) If $K = \mathbb{Q}$ and $g = 1$, then the conjecture holds.

(ii) ([18, Theorem 4]) If K is a quadratic number field other than the imaginary quadratic fields of class number one and $g = 1$, then the conjecture holds.

(iii) ([16, Corollary 4.5]) Let $\mathcal{A}(K, g, \ell)_{\mathrm{st}}$ be the set of K-isomorphism classes of abelian varieties in $\mathcal{A}(K, g, \ell)$ with semistable reduction everywhere. Then there exists an integer $C = C([K : \mathbb{Q}], g)$, depending only on $[K : \mathbb{Q}]$ and g, such that $\mathcal{A}(K, g, \ell)_{\mathrm{st}}$ is empty for any $\ell > C$ with $\ell \nmid d_K$. Here d_K is the discriminant of K.

(iv) ([2, Corollary 6.4] and [3]) Let K be a quadratic number field other than the imaginary quadratic fields of class number one. Let $\mathcal{A}(K, 2, \ell)_{\mathrm{QM}}$ be the set of K-isomorphism classes of QM-abelian surfaces by some quaternion division algebra over $\mathbb{Q}$ in $\mathcal{A}(K, 2, \ell)$. Then $\mathcal{A}(K, 2, \ell)_{\mathrm{QM}}$ is empty for any ℓ large enough.

For a g-dimensional abelian variety A over K, denote by $\rho_{A,\ell}\colon G_K \to GL(T_\ell(A)) \simeq GL_{2g}(\mathbb{Z}_p)$ the representation determined by the action of G_K on the ℓ-adic Tate module $T_\ell(A)$ of A. Consider the following conditions.

$(\mathrm{RT}_\ell)'$ For some finite extension L of K which is unramified at all places of K above ℓ, $L(A[\ell])$ is an ℓ-extension of $L(\mu_\ell)$.

$(\mathrm{RT}_{\mathrm{ab}})$ The representation $\rho_{A,\ell}$ has an abelian image.

It is clear that (RT_ℓ) implies $(\mathrm{RT}_\ell)'$. We recall the definitions of $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ and $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$.

Definition 3.2. We define the sets $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ and $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ of isomorphism classes of g-dimensional abelian varieties A over K as follows:
(1) $[A] \in \mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ if and only if A satisfies (RT_ℓ), $(\mathrm{RT}_{\mathrm{red}})$ and $(\mathrm{RT}_{\mathrm{ab}})$.
(2) $[A] \in \mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ if and only if A satisfies $(\mathrm{RT}_\ell)'$ and $(\mathrm{RT}_{\mathrm{ab}})$.

Clearly, we have $\mathcal{A}(K,g,\ell) \supset \mathcal{A}(K,g,\ell)_{\mathrm{ab}} \subset \mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$. Note that the reduction hypothesis $(\mathrm{RT}_{\mathrm{red}})$ is not imposed on abelian varieties in $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$. Hence $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ may be infinite (but the author does not know an example such that $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ is infinite).

4. Proof of Theorem 1.2

In this section, we use the same notation as in previous sections. First we study the structure of $A[\ell]$ for an abelian variety A in $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$. Let A be any g-dimensional abelian variety over K. We denote by $\bar{\rho}_{A,\ell}\colon G_K \to GL(A[\ell]) \simeq GL_{2g}(\mathbb{F}_p)$ the representation determined by the action of G_K on $A[\ell]$. Consider the following conditions.

$(\mathrm{RT}_{\mathrm{mod}})$ $(\bar{\rho}_{A,\ell})^{\mathrm{ss}}$ is conjugate to the direct sum of $2g$ characters which are of the form $\bar{\chi}_\ell^a$.

$(\mathrm{RT}_{\mathrm{mod}})'$ $(\bar{\rho}_{A,\ell})^{\mathrm{ss}}$ is abelian and the characters associated with $\bar{\rho}_{A,\ell}$ are of the form $\varepsilon\bar{\chi}_\ell^a$, where $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\ell^\times$ is a continuous character which is unramified at all places above ℓ.

The condition (RT_ℓ) is equivalent to the condition $(\mathrm{RT}_{\mathrm{mod}})$ by the lemma below. Hence the K-isomorphism class $[A]$ of g-dimensional abelian variety A over K is in $\mathcal{A}(K,g,\ell)$ if and only if A satisfies $(\mathrm{RT}_{\mathrm{mod}})$ and $(\mathrm{RT}_{\mathrm{red}})$.

Lemma 4.1. *Let A be a g-dimensional abelian variety over K.*

1. *The abelian variety A satisfies* (RT_ℓ) *if and only if A satisfies* $(\mathrm{RT}_{\mathrm{mod}})$.
2. *Suppose that $(\bar{\rho}_{A,\ell})^{\mathrm{ss}}$ is abelian. Then A satisfies* $(\mathrm{RT}_\ell)'$ *if and only if A satisfies* $(\mathrm{RT}_{\mathrm{mod}})'$.

Proof. The assertion (1) is proved by the arguments of the proof of [18, Lemma 3] and thus we omit the proof. Suppose that $(\bar{\rho}_{A,\ell})^{\mathrm{ss}}$ is abelian and denote by $\psi_1,\dots,\psi_{2g}$ the characters associated with $\bar{\rho}_{A,\ell}$. If A satisfies $(\mathrm{RT}_{\mathrm{mod}})'$, then we have $\psi_i = \varepsilon_i\bar{\chi}_\ell^{a_i}$ for some integer a_i where $\varepsilon_i\colon G_K \to \bar{\mathbb{F}}_\ell^\times$ is a continuous character which is unramified at all places of

K above ℓ. Let L be the composition field of all fields $\bar{K}^{\ker \varepsilon_i}$ for all i. Then L is unramified at all places of K above ℓ. Since each $\psi_i|_{G_{L(\mu_\ell)}}$ is trivial, we obtain $(\mathrm{RT}_\ell)'$. Conversely, suppose that $(\mathrm{RT}_\ell)'$ holds and take a field L as in the statement of $(\mathrm{RT}_\ell)'$. By (1), we know that each $\psi_i|_{G_L}$ is equal to $\bar{\chi}_\ell^{a_i}$ for some integer a_i. Hence $\varepsilon_i := \psi_i \cdot \bar{\chi}_\ell^{-a_i} \colon G_K \to \bar{\mathbb{F}}_\ell^\times$ is unramified at all places above ℓ and this implies $(\mathrm{RT}_{\mathrm{mod}})'$. □

We recall the following two propositions.

Proposition 4.2 (Faltings). *Fix an integer w. The set of isomorphism classes of semisimple n-dimensional ℓ-adic representations $G_K \to GL_n(\mathbb{Q}_\ell)$ which are $\mathbb{Q}$-integral with Frobenius weights less than or equal to w outside S, is finite.*

Proof. The Proposition follows from the proof of [7, Theorem 5]. See also [13, Chapter VIII, Section 5, Theorem 11]. □

Proposition 4.3 (Raynaud's criterion of semistable reduction, [9, Proposition 4.7]). *Suppose A is an abelian variety over a field F with a discrete valuation v, n is a positive integer not divisible by the residue characteristic, and the points of $A[n]$ are defined over an extension of F which is unramified over v. If $n \geq 3$ then A has semistable reduction at v.*

Here is one consequence of Proposition 4.3: Let A be an arbitrary abelian variety over a number field K. For any positive integer n, denote by $K(A[n])$ the field generated by K and the coordinates of all n-torsion points of A. Then A has semistable reduction everywhere over $K(A[12]) = K(A[3])K(A[4])$. In fact, the abelian variety A has semistable reduction over $K(A[3])$ outside 3 (apply Proposition 4.3 for $F = K(A[3])$) and has semistable reduction over $K(A[4])$ outside 2 (apply Proposition 4.3 for $F = K(A[4])$). For later use, we put $K(A[\ell^\infty]) = \bigcup_{n\geq 0} K(A[\ell^n])$ for any prime number ℓ.

For an integer $g > 0$, put

$$D_g := \sharp GL_{2g}(\mathbb{Z}/3\mathbb{Z}) \cdot \sharp GL_{2g}(\mathbb{Z}/4\mathbb{Z}).$$

If $\rho\colon G_K \to GL_{2g}(\mathbb{Q}_\ell)$ is an abelian representation, then, for any integer k, we denote by ρ^k the representation $G_K \to GL_{2g}(\mathbb{Q}_\ell)$ which is defined by $\rho^k(s) := (\rho(s))^k$ for any $s \in G_K$. With this notation, we obtain the following lemma which plays an important role in the proof of Theorem 1.2 to construct a good compatible system.

Lemma 4.4. *Let $g > 0$ be an integer and ℓ_0 a prime number. Let $\mathcal{A}_{\ell_0}$ be the set of isomorphism classes of representations $\rho\colon G_K \to GL_{2g}(\mathbb{Q}_{\ell_0})$ which are isomorphic to $\rho_{A,\ell_0}^{D_g}$ for some g-dimensional abelian variety A over K such that $K(A[\ell_0^\infty])$ is an abelian extension of K. Then $\mathcal{A}_{\ell_0}$ is finite.*

Proof. If A is an abelian variety over K such that $K(A[\ell_0^\infty])$ is an abelian extension of K, then A has potential good reduction everywhere (cf. [20, Section 2, Corollary 1]).

Putting $L := K(A[12])$, such an abelian variety A has good reduction everywhere over L by Proposition 4.3. Since $[L : K]$ divides D_g, the representation $\rho_{A,\ell_0}^{D_g}$ is unramified outside ℓ_0 for any g-dimensional abelian variety A over K such that $K(A[\ell_0^\infty])$ is an abelian extension of K. Take any finite place v of K not above ℓ_0. Let v_L be a finite place of L above v and denote by f the extension degree of $\mathbb{F}_{v_L}$ over $\mathbb{F}_v$, where $\mathbb{F}_{v_L}$ and $\mathbb{F}_v$ are residue fields of v_L and v, respectively. Remark that D_g/f is an integer since L is a Galois extension of K, and $\rho_{A,\ell_0}(\mathrm{Fr}_{v_L})$ is well-defined since A has good reduction everywhere over L (cf. [20, Theorem 1]). It is not difficult to obtain the equation

$$\det(XI_{2g} - \rho_{A,\ell_0}^{D_g}(\mathrm{Fr}_v)) = \det(XI_{2g} - (\rho_{A,\ell_0}(\mathrm{Fr}_{v_L}))^{D_g/f}).$$

Since A has good reduction everywhere over L, the polynomial $\det(XI_{2g} - \rho_{A,\ell_0}(\mathrm{Fr}_{v_L}))$ has rational integer coefficients and hence so is $\det(XI_{2g} - (\rho_{A,\ell_0}(\mathrm{Fr}_{v_L}))^{D_g/f})$. Consequently, the representation $\rho_{A,\ell_0}^{D_g}$ is $\mathbb{Q}$-integral with Frobenius weight $D_g/2$ outside the set of finite places of K above ℓ_0. Therefore, by Proposition 4.2, it is enough to prove that the representation $\rho_{A,\ell_0}^{D_g}$ is semisimple. Note that it has been already known that ρ_{A,ℓ_0} is semisimple (cf. [7, Theorem 3]). Since ρ_{A,ℓ_0} is abelian and geometric, the representation ρ_{A,ℓ_0} is locally algebraic in the sense of [19] (see also [8, Section 6, Proposition]). Therefore, by [17, (MT 1)], there exists a modulus of definition $\mathfrak{m}$ and an algebraic homomorphism $\phi\colon S_{\mathfrak{m}} \to GL_{2g}$ over $\mathbb{Q}$ such that the ℓ_0-adic representation induced by ϕ is isomorphic to ρ_{A,ℓ_0}. Here, the definition of the commutative algebraic group $S_{\mathfrak{m}}$ over $\mathbb{Q}$ is given in [19, Chapter II]. Note that any ℓ_0-adic representation arising from an algebraic morphism $S_{\mathfrak{m}} \to GL_{2g}$ is automatically semisimple. Since $\rho_{A,\ell_0}^{D_g}$ comes from the composition $S_{\mathfrak{m}} \overset{D_g}{\to} S_{\mathfrak{m}} \overset{\phi}{\to} GL_{2g}$ where $S_{\mathfrak{m}} \overset{D_g}{\to} S_{\mathfrak{m}}$ is the multiplication by D_g map, we obtain the fact that $\rho_{A,\ell_0}^{D_g}$ is semisimple. □

Proof of Theorem 1.2. First we note that, if an abelian variety A over K satisfies $(\mathrm{RT}_{\mathrm{ab}})$, then $\rho_{A,\ell'}$ is abelian for any prime number ℓ' (cf. [19, Chapter III, Section 2.3, Corollary 1]). Fix a prime number ℓ_0 and denote by $\mathcal{A}_{\ell_0}$ the set as in Lemma 4.4. Assume that there exist infinitely many prime numbers ℓ such that $\mathcal{A}'(K, g, \ell)_{\mathrm{ab}}$ is not empty. For every such ℓ, we obtain the ℓ_0-adic representation $\rho_{A,\ell_0}^{D_g}$ which is in the set $\mathcal{A}_{\ell_0}$, where A is an abelian variety whose isomorphism class is in the set $\mathcal{A}'(K, g, \ell)_{\mathrm{ab}}$. By Lemma 4.4, we see that there exists a representation ρ_{ℓ_0} in $\mathcal{A}_{\ell_0}$ satisfying the following: For infinitely many ℓ, there exists an element $[A] \in \mathcal{A}'(K, g, \ell)_{\mathrm{ab}}$ such that the representation $\rho_{A,\ell_0}^{D_g}$ is isomorphic to ρ_{ℓ_0}. Thus we know that the representation ρ_{ℓ_0} extends to a $\mathbb{Q}$-integral strictly compatible system $(\rho_\ell)_\ell$ of $2g$-dimensional abelian semisimple ℓ-adic representations of G_K. Furthermore, for infinitely many prime numbers ℓ, the characters associated with a residual representation $\bar{\rho}_\ell$ of ρ_ℓ are of the form $\varepsilon\bar{\chi}_\ell^a$ by Lemma 4.1, where $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\ell^\times$ is a continuous character which is unramified at all places of K above ℓ. Applying Corollary 2.12, we see that there exist integers $m_1, \ldots, m_{2g}$ and a finite extension L of K such that

ρ_{ℓ_0} is isomorphic to $\chi_{\ell_0}^{m_1} \oplus \chi_{\ell_0}^{m_2} \oplus \cdots \oplus \chi_{\ell_0}^{m_{2g}}$ on G_L. In particular, for some prime number ℓ and some $[A] \in \mathcal{A}'(K, g, \ell)_{\rm ab}$, $\rho_{A,\ell_0}^{D_g}$ is isomorphic to $\chi_{\ell_0}^{m_1} \oplus \chi_{\ell_0}^{m_2} \oplus \cdots \oplus \chi_{\ell_0}^{m_{2g}}$ on G_L. Therefore, looking at the eigenvalues of the images of a Frobenius element (at some place) of $\rho_{A,\ell_0}^{D_g}|_{G_L}$ and $\chi_{\ell_0}^{m_1} \oplus \chi_{\ell_0}^{m_2} \oplus \cdots \oplus \chi_{\ell_0}^{m_{2g}}$, we know that $D_g/2 = m_1 = m_2 = \cdots = m_{2g}$. Since $\rho_{A,\ell_0}^{D_g}|_{G_L}$ has Hodge-Tate weights 0 and D_g at a place of L above ℓ_0 (cf. [19, III-7]), this is a contradiction. □

References

[1] N. Bourbaki, Algèbre. Chapitre 8, Éléments de mathématique. 23. Première partie: Les structures fondamentales de l'analyse. Livre II: Actualités Sci. Ind. no. 1261, Hermann, Paris, 1958.

[2] K. Arai, Galois images and modular curves, Algebraic Number Theory and Related topics 2010 – a volume in RIMS-Bessatsu Series –.

[3] K. Arai and F. Momose, Algebraic points on Shimura curves of $\Gamma_0(p)$-type, available at http://arxiv.org/abs/1202.4841

[4] X. Caruso, Représentations semi-stables de torsion dans le case $er < p-1$, J. Reine Angew. Math. **594** (2006), 35–92.

[5] P. Deligne, La conjecture de Weil I, Publ. Math. Inst. Hautes Études Sci. **43** (1974), 273–308.

[6] P. Deligne, La conjecture de Weil II, Publ. Math. Inst. Hautes Études Sci. **52** (1980), 137–252.

[7] G. Faltings, Finiteness theorems for abelian varieties over number fields, Arithmetic Geometry (Ed. Gary Cornell and Joseph H. Silverman), 9–27, Springer-Verlag, 1986.

[8] J.-M. Fontaine and B. Mazur, Geometric Galois Representations, Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), 41–78, Ser. Number Theory I, Int. Press, Cambridge, MA, 1995.

[9] A. Grothendieck, "Modèles de Néron et monodromie, in Groupes de monodromie en géometrie algébrique, SGA 7", Lecture Notes in Mathematics **288**, 313–523, Berlin, New York, Springer-Verlag, 1972.

[10] C. Khare, Compatible system of mod p Galois representations and Hecke characters, Math. Res. Lett. **10** (2003), 71–83.

[11] C. Khare, Reciprocity law for compatible systems of abelian mod p Galois representations, Canad. J. Math. **57** (2005), 1215–1223.

[12] M. Kisin and G. Lehrer, Eigenvalues of Frobenius and Hodge numbers, Pure Appl. Math. Q. **2** (2006), 497–518.

[13] S. Lang, Algebraic number theory, Second edition, Graduate Texts in Mathematics **110**, Springer-Verlag, 1994.

[14] B. Mazur, Rational isogenies of prime degree, Invent. Math. **44** (1978), 129–162.

[15] F. Momose, Isogenies of prime degree over number fields, Compos. Math. **97** (1995), 329–348.

[16] Y. Ozeki, Non-existence of certain Galois representations with a uniform tame inertia weight, Int. Math. Res. Not. IMRN **2011** (2011), 2377–2395.

[17] K. A. Ribet, Galois action on division points of Abelian varieties with real multiplications, Amer. J. Math. **98** (1976), 751–804.

[18] C. Rasmussen and A. Tamagawa, A finiteness conjecture on abelian varieties with constrained prime power torsion, Math. Res. Lett. **15** (2008), 1223–1231.

[19] J.-P. Serre, Abelian l-adic representations and elliptic curves, With the collaboration of Willem Kuyk and John Labute, Second edition, Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.

[20] J.-P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. (2) **8** (1986), 492–517.

[21] Y. Taguchi, On potentially Abelian geometric representations, Ramanujan J. **7** (2003), 477–483.

[22] A. Weil, On a certain type of characters of the idele-class group of an algebraic number field, Proceedings of the international symposium on algebraic number theory, Tokyo and Nikko, 1955, pp. 1–7. Science Council of Japan, Tokyo, 1956.

[23] Y. G. Zarhin, A finiteness theorem for unpolarized abelian varieties over number fields with prescribed places of bad reduction, Invent. Math. **79** (1985), 309–321.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan
E-mail address: yozeki@kurims.kyoto-u.ac.jp