Non-existence results on certain Abelian varieties

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This thesis is dedicated to my parents
Matsuo Ozeki and Masuko Ozeki.

Contents

Introduction

The aim of this thesis is to study a conjecture of Rasmussen and Tamagawa ([**RT**]) which is related with the non-existence of certain abelian varieties with constrained prime power torsion subgroups, and to give some results on some variants of this conjecture. Problems of non-existence or finiteness for isomorphism classes of various abelian varieties have been studied by many mathematicians. As some famous results on such problems, we know

- the Shafarevich Conjecture, proved by Faltings in [**Fa**], which is as follows: there exist only finitely many isomorphism classes of abelian varieties over a given number field, with polarization of a given degree, which have good reduction outside given places. Furthermore, Zarhin [**Za**] improved on Fartings' result by omitting the assumption about polarization. We may say that this is an analogue of Hermite-Minkowski theorem (there exist only finitely many isomorphism classes of number fields with given degree and ramification set of places), which is also famous number theoretic result.
- Fontaine's theorem in [**Fo2**] by a study of ramification theory. He showed that there exist no abelian varieties over the rational number field with everywhere good reduction.

A conjecture of Rasmussen and Tamagawa is in the spirit of the Shafarevich Conjecture. Rather than fixing a specific reduction type for an abelian variety, they varied it in some special conditions and placed an arithmetic constraint on torsion. More precisely, Rasmussen and Tamagawa defined the set $\mathcal{A}(K, g, \ell)$ of g-dimensional isomorphism classes of certain abelian varieties over K with constrained prime power torsion. We can see easily that the set $\mathcal{A}(K, g, \ell)$ is finite by the Shafarevich Conjecture as above. Rasmussen and Tamagawa conjectured that such a finiteness should hold *if we take the union of these sets for ℓ varies over all primes.*

CONJECTURE ([**RT**], Conjecture 1). The set

$$\mathcal{A}(K, g) := \{(A, \ell) \mid [A] \in \mathcal{A}(K, g, \ell),\ \ell : \text{prime number}\}$$

is finite, that is, the set $\mathcal{A}(K, g, \ell)$ is empty for any prime ℓ large enough.

This conjecture is shown only in a few case in published papers, see Theorem 5.5. Rasmussen and Tamagawa showed the conjecture in the case where $g \leq 3$ and K is a certain small number field. Under the generalized Riemann hypothesis, Rasmussen and Tamagawa proved this conjecture (but it is unpublished, yet). We prove the following results on variants of the Rasmussen-Tamagawa Conjecture (but statements here are rough. See Chapter 6 and 7 for precise statements).

THEOREM. (1) (*Corollary 6.15*) *There exists a constant C which depends only on K and g such that the set*

$$\mathcal{A}(K, g, \ell) \cap \{\textit{semistable reduction everywhere}\}$$

is empty for any prime number $\ell > C$. In fact, we may take $C = 2^{\delta_1}\binom{2g}{g}$, where d is the extension degree of $K/\mathbb{Q}$, d_K the discriminant of K and $\delta_1 := 2dg + 1$.
(2) (*Theorem 7.1*) *There exists a constant C which depends only on K and g such that the set*

$$\mathcal{A}(K, g, \ell) \cap \{\textit{whose } \ell\textit{-adic representation has an abelian image}\}$$

is empty for any prime number $\ell > C$.

Now we describe the organization of this paper. In Chapter 1 to 4, we recall basic theories which we use in this paper. In Chapter 5, we explain the Conjecture of Rasmussen and Tamagawa. In Chapter 6, we prove the non-existence of certain Galois representations and prove (1) of the above theorem. Finally in Chapter 7, we prove (2) of the above theorem.

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Notation

Throughout this paper, we use the following notations:

- $\mathbb{Q}$: the rational number field.
- $\mathbb{Z}$: the ring of rational integers.

For any prime number ℓ,

- $\mathbb{Q}_\ell$: the ℓ-adic completion of $\mathbb{Q}$.
- $\mathbb{Z}_\ell$: the valuation ring of $\mathbb{Q}_\ell$.
- $\mathbb{F}_\ell$: the finite field of ℓ-elements.

For any field F, we choose an algebraic closure $\bar{F}$ of F and

- F^{sep} : the separable closure of F in $\bar{F}$.
- $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$: the absolute Galois group of F.

Except Chapter 3, we always use K to denote a number field, that is, a finite extension of $\mathbb{Q}$ and

- K^{ab} : the maximal abelian extension of K in $\bar{K}$.
- $G_K^{\mathrm{ab}} := \mathrm{Gal}(K^{\mathrm{ab}}/K)$: the Galois group of K^{ab} over K.
- I_K : the idele group of K.

For a finite place v of K,

- K_v : the completion of K at v with integer ring $\mathcal{O}_v$.
- $U_v := \mathcal{O}_v^\times$: the group of units of $\mathcal{O}_v$.
- $G_v := \mathrm{Gal}(\bar{K}_v/K_v)$: the absolute Galois group of K_v.
- $I_v \subset G_v$: the inertia subgroup of v.
- q_v : the order of the residue field of v.

Fixing an embedding $\bar{K} \hookrightarrow \bar{K}_v$ (or equivalently, choosing an extension of v to $\bar{K}$), we identify G_v (resp. I_v) with a decomposition group of K at v (resp. an inertia subgroup of K at v).

For any scheme X over a commutative ring R and an R-algebra R', we denote the fiber product $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$ by $X_{R'}$.

CHAPTER 1

Galois representations

In this chapter, we recall some basic notions of Galois representations. Throughout this chapter, we write K for a finite extension of $\mathbb{Q}$.

1. Definition

Let G be a Galois group.

Definition 1.1. A *Galois representation* (defined over A) is a continuous homomorphism

$$\rho\colon G \to GL_n(A),$$

where A is some topological ring and n is a positive integer. Two Galois representations ρ_1 and ρ_2 are *equivalent (or isomorphic)* if there exists a matrix $P \in GL_n(A)$ such that $P^{-1}\rho_1 P = \rho_2$. We call n the *degree* (or, *dimension*) *of* ρ.

Given such a thing, we can consider the free A-module A^n of rank n together with a continuous action of G by defining $\sigma.m = \rho(\sigma)m$. Conversely, given a finite free A-module M of rank n with a continuous A-linear action of G, we obtain a representation ρ as above by choosing a basis for M. Changing the basis yields an equivalent representation. Therefore, to give a Galois representation $G \to GL_n(A)$ is the same as to give a finite free A-module of rank n with continuous action of G.

Definition 1.2. (1) A Galois representation $\rho\colon G \to GL_n(A)$ is called *abelian* if $\mathrm{Im}(\rho)$ is an abelian group.
(2) A Galois representation $\rho\colon G \to GL_n(A)$ is called *potentially abelian* if $\rho|_H$ is abelian for some finite index subgroup H of G.

Clearly, any 1-dimensional Galois representation is abelian.

Definition 1.3. A Galois representation is called Artinian if it has finite image.

Let ℓ be a prime number and λ a finite place of a number field E. We denote by E_λ the completion of E at λ and $\mathbb{F}_\lambda$ the residue field of λ.

Definition 1.4. (1) If A is the field $\mathbb{Q}_\ell$ (resp. E_λ), then a Galois representation defined over A is called an *ℓ-adic* (resp. *λ-adic*) *representation*.

(2) If A is the subfield of $\bar{\mathbb{F}}_\ell$, then a Galois representation defined over A is called a *mod ℓ representation*. We call a Galois representation defined over $\mathbb{F}_\lambda$ a *mod λ representation*.

DEFINITION 1.5. For a Galois representation V of G (defined over A), put

$$V^\vee := \mathrm{Hom}_A(V, A)$$

and equip $V^\vee$ with the G-action defined by $g.f(v) := f(g^{-1}.v)$ for $f \in V^\vee, g \in G$ and $v \in V$. We call $V^\vee$ the *dual of* V.

REMARK 1.6. Let $\rho\colon G \to GL_A(V) \simeq GL_n(A)$ be a Galois representation (for certain basis of V). Then we can choose a natural basis of $V^\vee$ such that corresponding representation $\rho^\vee\colon G \to GL_A(V^\vee) \simeq GL_n(A)$ is given by $\rho^\vee(g) = (\rho(g)^t)^{-1}$. Here $\rho(g)^t$ is the transposed matrix of $\rho(g)$.

DEFINITION 1.7. (1) Let F be a complete discrete valuation field with valuation v, I the inertia subgroup of G_F and I_w the wild inertia subgroup of G_F, that is, the maximal pro-ℓ subgroup of I where ℓ is the residue characteristic of v. Let $\rho\colon G_F \to GL_n(A)$ be a Galois representation. We say that ρ is *unramified* (resp. *tamely ramified*) if $\rho|_I$ (resp. $\rho|_{I_w}$) is trivial. We say that ρ is *wildly ramified* if ρ is not tamely ramified.
(2) Let $\rho\colon G_K \to GL_n(A)$ be a Galois representation of G_K. Let v be a finite place of K. Then we say that ρ is *unramified at v* (resp. *tamely ramified at v*) if the restriction of ρ to $G_v = G_{K_v}$ is unramified (resp. tamely ramified). If ρ is unramified at v, the notion $\rho(\mathrm{Fr}_v) \in GL_n(A)$ is well-defined by a natural manner, where Fr_v is the arithmetic Frobenius of v. We say that ρ is *wildly ramified at v* if ρ is not tamely ramified at v.
(3) Let $\rho\colon G_K \to GL_n(A)$ be a Galois representation of G_K. Let S be a finite set of places of K. We say that ρ is *unramified outside S* (*or away from S*) if ρ is unramified at all finite places of K not in S.

EXAMPLE 1.8. Suppose $G = G_F$, where F is a field whose characteristic is prime to ℓ. For any positive integer i, we choose a primitive ℓ^i-th root of unity ζ_{ℓ^i} in $\bar{F}$ such that $\zeta_{\ell^{i+1}}^\ell = \zeta_{\ell^i}$. Then there exists an integer $a_g(i)$ such that $g(\zeta_{\ell^i}) = \zeta_{\ell^i}^{a_g(i)}$ for $g \in G_F$. Note that ℓ does not divide $a_g(i)$. Since $g(\zeta_{\ell^{i+1}})^\ell = g(\zeta_{\ell^i})$, we obtain $a_g(i+1) \equiv a_g(i) \bmod \ell^i$. Hence there exists a unique ℓ-adic unit $\chi_\ell(g) \in \mathbb{Z}_\ell^\times$ such that $\chi_\ell(g) \equiv a_g(i) \bmod \ell^i$ for all i, and we obtain 1-dimensional Galois representation $\chi_\ell\colon G_F \to \mathbb{Z}_\ell^\times$, which is called an *$\ell$-adic cyclotomic character*. The representation $\bar{\chi}_\ell\colon G_F \to \mathbb{F}_\ell^\times$ defined by $\bar{\chi}_\ell(g) := (\chi_\ell(g) \bmod \ell) = (a_g(1) \bmod \ell)$ is called a *mod ℓ cyclotomic character*.

Let E be a finite extension of $\mathbb{Q}$ and λ a finite place of E above ℓ. The *λ-adic cyclotomic character* χ_λ and the *mod λ cyclotomic character* $\bar{\chi}_\lambda$ are characters $G_F \xrightarrow{\bar{\chi}_\ell} \mathbb{Q}_\ell^\times \hookrightarrow \mathbb{E}_\lambda^\times$ and $G_F \xrightarrow{\bar{\chi}_\ell} \mathbb{F}_\ell^\times \hookrightarrow \mathbb{F}_\lambda^\times$, respectively.

If F is a number field K, it is not difficult to check that χ_ℓ and $\bar{\chi}_\ell$ are unramified outside ℓ and, for any finite place v of K not above ℓ, we have $\chi_\ell(\mathrm{Fr}_v) = q_v$ and $\bar{\chi}_\ell(\mathrm{Fr}_v) = (q_v \bmod \ell)$.

For any integer i, we denote by $\mathbb{Z}_\ell(i)$ the Galois representation of G_F which is $\mathbb{Z}_\ell$ as a set and equipped with G_F-action defined by $g.x := \chi_\ell^i(g)x$. For any Galois representation V defined over a $\mathbb{Z}_\ell$-algebra A, we put $V(i) := V \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(i)$ and equip $V(i)$ with a natural G_F-action. We call $V(i)$ the *i-th Tate twist of V*.

2. Geometric Galois representations

Let ℓ be a prime number.

Definition 1.9. (1) An ℓ-adic Galois representation $\rho\colon G_K \to GL_n(\mathbb{Q}_\ell)$ is called *geometric* if

(a) it is unramified outside a finite set of places of K;
(b) its restriction to every decomposition group G_v (for v running through all finite places of K) is potentially semistable in the sense of Fontaine (for the places above p) and Grothendieck[1] (for the places not above p).

(2) Let E be a finite extension of $\mathbb{Q}_\ell$. An E-representation of degree n of G_K is called *geometric* if it is geometric as a ℓ-adic representation of degree $n \cdot [E : \mathbb{Q}]$.
(3) Let E be an algebraic extension of $\mathbb{Q}_\ell$. An E-representation V of degree n of G_K is called *geometric* if there exist a finite extension E_0 of ℓ in E, a geometric E_0 representation V_0 of G_K and an isomorphism of E-representations $E \otimes_{E_0} V_0 \simeq V$.

By Grothendieck's monodromy theorem, the condition (b) in (1) is equivalent to say the condition (b)′ below:

(b)′ its restriction to decomposition group G_v (for v running through all finite places of K above ℓ) is potentially semistable.

Let X be a proper smooth variety over K. Let $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ be the ℓ-adic étale cohomology group of X. Then $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is a $\mathbb{Q}_\ell$-vector space of dimension $b_r(X)$, where $b_r(X)$ is the r-th Betti number of $X(\mathbb{C})$. It is known that $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ has a natural G_K-action and thus it is a Galois representation of G_K. For a finite place v of K such that X_{K_v} is a generic fiber of a proper smooth scheme over $\mathcal{O}_v$, it is known that $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unramified at v. Since there exists infinitely many such v, we know that $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ ramifies only finitely many v. Moreover, it is known that $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is geometric, which is a deep result of Tsuji [**Ts2**]. An irreducible $\mathbb{Q}_\ell$-representation of G_K is said to *come from algebraic geometry* if it is isomorphic to a subquotient

[1] An ℓ-adic representation V of G_K is *semistable* at a place v not above ℓ if I_v acts on V unipotently. An ℓ-adic representation V of G_K is *potentially semistable* at a place v not above ℓ if some open subgroup of I_v acts on V unipotently.

of an étale cohomology group $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell(i))$ with coefficients in $\mathbb{Q}_\ell(i)$ for some integer i, of a proper smooth algebraic variety X over K.

The following amazing conjecture is well-known.

CONJECTURE 1.10. *An irreducible ℓ-adic representation of G_K is geometric if and only if it comes from algebraic geometry.*

The part of Conjecture 1.10 saying that irreducible "geometric" representations "come from algebraic geometry" is known for irreducible potentially abelian representations by the following well-known result (however, we write this result without giving definition of some notions).

PROPOSITION 1.11 ([**FM**], Part 1, Section 6). *Let $\rho\colon G_K \to GL_n(\mathbb{Q}_\ell)$ be a potentially abelian ℓ-adic Galois representation of G_K. Then the following are equivalent:*
(1) *ρ is locally algebraic* (*see Section 3*).
(2) *ρ is Hodge-Tate at all finite places v above ℓ.*
(3) *ρ is de Rham at all finite places v above ℓ.*
(4) *ρ is potentially semistable at all finite places v above ℓ.*
(5) *ρ is potentially crystalline at all finite places v above ℓ.*
(6) *ρ is geometric.*
(7) *ρ comes from CM abelian varieties* (*up to a finite image*).

3. Artin conductor

The Artin conductor is an invariant which measures the depth of ramifications of Galois representations (cf. [**Se1**], Chapter VI or [**Se2**], Chapter 19). Let F be a field. Let $\rho\colon G_K \to GL(V) \simeq GL_d(F)$ be a Galois representation of a number field K with finite image. Here we equip $GL_d(F)$ with the discrete topology. We define its *Artin conductor* $\mathfrak{N}(\rho)$ as follows: Choose a finite Galois extension L over K such that ρ factors through $\mathrm{Gal}(L/K)$ and define

$$\mathfrak{N}(\rho) := \prod_{\mathfrak{p}} \mathfrak{p}^{n(\mathfrak{p},\rho)},$$

where $\mathfrak{p}$ runs through the nonzero prime ideals of K not dividing the characteristic of F and, for each $\mathfrak{p}$,

$$\begin{aligned} n(\mathfrak{p},\rho) : &= \sum_{i=0}^{\infty} \frac{1}{[G_0 : G_i]} \dim_F (V/V^{G_i}) \\ &= \int_{-1}^{\infty} \dim_F V/V^{G^u} du \end{aligned}$$

where G_i (resp. G^u) is the i-th ramification subgroup of $\mathrm{Gal}(L/K)$ in lower numbering (resp. the u-th ramification subgroup of $\mathrm{Gal}(L/K)$ in upper numbering)[2] of the decomposition group of a prime of L lying above $\mathfrak{p}$, and V^{G_i} is the fixed part of V by the action of G_i. If F is a field of characteristic ℓ, then the Artin conductor of ρ is said to be the *Artin conductor outside* ℓ. Since the ramification filtration in upper numbering is compatible with taking quotients, the exponent $n(\mathfrak{p},\rho)$ does not depend on the choice of L. It is known that $n(\mathfrak{p},\rho)$ is a non-negative integer. It is clear that $n(\mathfrak{p},\rho)>0$ if and only if ρ ramifies at $\mathfrak{p}$. Put

$$\begin{aligned}\mathrm{sw}(\mathfrak{p},\rho) :&= \sum_{i=1}^{\infty}\frac{1}{[G_0:G_i]}\dim_F(V/V^{G_i})\\ &= \int_0^{\infty}\dim_F V/V^{G^u}du.\end{aligned}$$

Then ρ is wildly ramified at $\mathfrak{p}$ if and only if $\mathrm{sw}(\mathfrak{p},\rho)>0$. By definition, if ρ is of dimension 1, we see that $\mathrm{sw}(\mathfrak{p},\rho)$ is the minimal $u\geq 0$ such that G^u acts on V trivially.

Example 1.12 (1-dimensional case). Suppose that F is a field of characteristic $\ell>0$. Let $\rho\colon G_K\to F^\times$ be a representation of dimension 1 and $\mathfrak{p}$ a prime ideal of K different from ℓ. Let $K_{\mathfrak{p}}$ be the completion of K at $\mathfrak{p}$. Denote by $U_{\mathfrak{p}}^{(0)}$ the group of units of the integer ring $\mathcal{O}_{\mathfrak{p}}$ of $K_{\mathfrak{p}}$ and, for any positive integer $u>0$, put $U_{\mathfrak{p}}^{(u)} := 1+\mathfrak{p}^u$. By class field theory, we may regard ρ as a representation $I_K\to F^\times$ of the idele group of K and may regard $\rho|_{G_{\mathfrak{p}}}$ as a representation $K_{\mathfrak{p}}^\times\to F^\times$ of $K_{\mathfrak{p}}^\times$ where $G_{\mathfrak{p}}$ is a decomposition subgroup of G_K at $\mathfrak{p}$. By using the Hasse-Arf Theorem, we see that $n(\mathfrak{p},\rho)$ is the minimal integer $u\geq 0$ such that $\rho(U_{\mathfrak{p}}^{(u)})=1$.

Furthermore suppose that F is a subfield of $\bar{\mathbb{F}}_\ell$. Let $\mathfrak{N}=\mathfrak{N}(\rho)$ be the Artin conductor of ρ outside ℓ. Then we can show that the fixed field of the kernel of ρ is contained in the strict ray class field of K of conductor $\mathfrak{N}(\rho)\ell$. This can be checked as follows: For any ideal $\mathfrak{a}$ of the integer ring of K, put $U_{\mathfrak{p}}(\mathfrak{a}) := \ker(\mathcal{O}_{\mathfrak{p}}^\times\to(\mathcal{O}_{\mathfrak{p}}/\mathfrak{a}\mathcal{O}_{\mathfrak{p}})^\times)$ and denote by $K(\mathfrak{a})$ the strict ray class field of K of conductor $\mathfrak{a}$. Then $U_{\mathfrak{p}}(\mathfrak{N})$ is contained in the kernel of ρ for any prime ideal $\mathfrak{p}$ of K different from ℓ by the last sentence of the first paragraph. Choose an integer $m\geq 0$ large enough such that ρ is trivial on $U_{\mathfrak{p}}(\ell^m)$ for any prime ideals $\mathfrak{p}$ of K above ℓ. Then $\prod_{\mathfrak{p}}U_{\mathfrak{p}}(\mathfrak{N}\ell^m)\times(K_\infty^\times)^0\subset I_K$ is contained in the kernel of ρ. Here $(K_\infty^\times)^0$ is the connected component of the identity of $K_\infty^\times$ where K_∞ is the product of the completions of K at the archimedean places. Class field theory says that $\rho\colon G_K\to F^\times$ must factor through $\mathrm{Gal}(K(\mathfrak{N}\ell^m)/K)$. Since $F^\times$ is prime-to-ℓ, ρ in fact factors through $\mathrm{Gal}(K(\mathfrak{N}\ell)/K)$.

[2]In particular, for a chosen prime of L lying above $\mathfrak{p}$, $G_{-1}=G^{-1}$ is the decomposition group, $G_0=G^0$ is an inertia subgroup and G_1 is a wild inertia subgroup.

Let ρ be an abelian and semisimple representation of G_K of arbitrary dimension, then the fixed field of the kernel of ρ is in fact contained in the ray class field of conductor $\mathfrak{N}(\rho)\ell$. Indeed, such a representation ρ is isomorphic to a direct summand of 1-dimensional representations over $\bar{\mathbb{F}}_\ell$ by Schur's lemma[3].

4. Compatible systems

Let E be a finite extension of $\mathbb{Q}$. For a finite place λ of E, we denote by ℓ_λ the prime number below λ, E_λ the completion of E at λ and $\mathbb{F}_\lambda$ the residue field of λ. We denote by E_λ the completion of E at a finite place λ of E. Let S be a finite set of finite places of K and T a finite set of finite places of E. Put $S_\ell := S \cup \{\text{places of } K \text{ above } \ell\}$. A representation $\rho\colon G_K \to GL_n(E_\lambda)$ is said to be *E-rational with ramification set S* if ρ is unramified outside S_ℓ and the characteristic polynomial $\det(T - \rho(\mathrm{Fr}_v))$ of Fr_v has coefficients in E for each finite place $v \notin S_\ell$ of K, where Fr_v is an arithmetic Frobenius of v.

Now we give definitions of compatible systems of λ-adic (resp. mod λ) representations, which mainly follows from that in [**Kh1**] and [**Kh2**]. An *E-rational strictly compatible system $(\rho_\lambda)_\lambda$ of n-dimensional λ-adic representations of G_K with defect set T and ramification set S*, consists of, for each finite place λ of E not in T, a continuous representation $\rho_\lambda\colon G_K \to GL_n(E_\lambda)$ that is

(i) ρ_λ is unramified outside S_{ℓ_λ};
(ii) for any finite place $v \notin S$ of K, there exists a monic polynomial $f_v(T) \in E[T]$ such that for all places $\lambda \notin T$ of E which is coprime to the residue characteristic of v, the characteristic polynomial $\det(T - \rho_\lambda(\mathrm{Fr}_v))$ of Fr_v is equal to $f_v(T)$.

An *E-rational strictly compatible system $(\bar{\rho}_\lambda)_\lambda$ of n-dimensional mod λ representations of G_K with defect set T and ramification set S*, consists of, for each finite place λ of E not in T, a continuous representation $\bar{\rho}_\lambda\colon G_K \to GL_n(\mathbb{F}_\lambda)$ that is

(i) $\bar{\rho}_\lambda$ is unramified outside S_{ℓ_λ};
(ii) for any finite place $v \notin S$ of K, there exists a monic polynomial $f_v(T) \in E[T]$ such that for all places $\lambda \notin T$ of E which is coprime to the residue characteristic of v, $f_v(T)$ is integral at λ and the characteristic polynomial $\det(T - \rho_\lambda(\mathrm{Fr}_v))$ of Fr_v is the reduction of $f_v(T)$ mod λ.

We will often suppress the sets S and T from the notations.

Example 1.13. Let X be a proper smooth variety over K. Let $V_\ell := H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)^\vee$ be the dual of the ℓ-adic étale cohomology group $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ of X. Then the system $(V_\ell)_\ell$ is a strict compatible system

[3]It follows from Schur's lemma that an irreducible abelian representation of a finite group defined over an algebraically closed field is of dimension 1.

whose defect set is all prime numbers and ramification set is the set of finite places v of K such that X has bad reduction at v. This fact follows from the Weil Conjecture which is proved by Deligne (cf. [**De1**], [**De2**].

It is conjectured that every E-rational strictly compatible system arises motivically.

CONJECTURE 1.14 ([**Kh1**], Conjecture 1). *Any E-rational strictly compatible system of λ-adic (resp. mod λ) representations arises motivically*[4].

CONJECTURE 1.15 ([**Kh1**], Conjecture 2). *Let $(\rho_\lambda)_\lambda$ be an E-rational strictly compatible system of mod λ representations with defect set T and ramification set S.*
(1) (*Lifting*) *It lifts to (i.e., is the reduction up to semisimplification of) a strictly compatible system of semisimple λ-adic representations.*
(2) (*Bounded conductor*) *It is of bounded Artin conductor (the definition of this notion is below).*
(3) (*Purity*) *Assume that ρ_λ is irreducible for almost all λ. Then the roots of $f_v(X)$ for finite places v not in S are of absolute value q_v^t with respect to all embeddings of $\bar{\mathbb{Q}}$ in $\mathbb{C}$, and for an half integer t that is independent of v.*
(4) (*Integrality*) *$(\rho_\lambda \otimes \bar{\chi}_\lambda^m)_\lambda$ is integral where $\bar{\chi}_\lambda$ is the mod λ cyclotomic character and m is some integer.*

Because of known properties of Galois representations which arise from geometry, one expects that Conjecture 1.14 implies Conjecture 1.15. These conjectures hold for abelian semisimple strictly compatible systems (see Theorem 1.21).

An *inertial level $\mathfrak{L}$ of K* is a collection $(\mathfrak{L}_v)_v$ of open normal subgroups $\mathfrak{L}_v$ of I_v for each finite place v of K such that $\mathfrak{L}_v = I_v$ for almost all v. An *inertial level $\mathfrak{L}$ of a geometric λ-adic representation ρ_λ of G_K* is the collection $(\mathfrak{L}_v(\rho_\lambda))_v$ of open normal subgroups $\mathfrak{L}_v(\rho_\lambda)$ of I_v for each finite place v of K, where $\mathfrak{L}_v(\rho_\lambda)$ is the largest open subgroup of I_v such that the restriction of ρ_λ to $\mathfrak{L}_v(\rho_\lambda)$ is semistable. By definition, we have $\mathfrak{L}_v(\rho_\lambda) = I_v$ for almost all v. A compatible system $(\rho_\lambda)_\lambda$ of geometric λ-adic representations of G_K has a *bounded inertial level* if there exists an inertial level $\mathfrak{L} = (\mathfrak{L}_v)_v$ such that $\mathfrak{L}_v \subset \mathfrak{L}_v(\rho_\lambda)$ for all λ and v. A λ-adic representation ρ_λ is *E-rational with Frobenius weights $w_1, w_2, \ldots, w_n$ outside S* if ρ_λ is E-rational with ramification set S and for all finite places $v \notin S_\ell$ of K, the complex roots of the characteristic polynomial $\det(T - \rho(\mathrm{Fr}_v))$ of Fr_v (, for a chosen embedding of

[4]In the article [**Kh1**], Conjecture 1.14 (in this paper) is written only for *E-integral mod λ* representations. However, we may extend this condition "E-integral" to "E-rational" by Conjecture 1.15, and it is not difficult to see that "mod λ case" implies "λ-adic case".

E into $\mathbb{C}$,) have their complex absolute values $q_v^{w_1/2}, q_v^{w_2/2}, \ldots, q_v^{w_n/2}$ where q_v is the cardinality of the residue field of v. A strict compatible system $(\rho_\lambda)_\lambda$ is said to be *E-rational strict compatible system with Frobenius weights* $w_1, w_2, \ldots, w_n$ if each ρ_λ is E-rational with Frobenius weights $w_1, w_2, \ldots, w_n$ outside a ramification set of $(\rho_\lambda)_\lambda$. We call $w_1, w_2, \ldots, w_n$ the Frobenius weights of ρ_λ (resp. $(\rho_\lambda)_\lambda$) and ρ_λ (resp. $(\rho_\lambda)_\lambda$) is said to be *pure* if $w_1 = w_2 = \cdots = w_n$. Finally, a compatible system $(\rho_\lambda)_\lambda$ of geometric λ-adic representations of G_K has *bounded Hodge-Tate weights* if there exist integers a and b with $a \le b$ such that, for any λ and finite place v of K above ℓ_λ, all the Hodge-Tate weights of $\rho|_{G_v}$ viewed as a $\mathbb{Q}_\ell$-representation are in $[a, b]$.

Finally, a compatible system $(\bar\rho_\lambda)_\lambda$ of mod λ representations of G_K is of *bounded Artin conductor* if there exists an ideal $\mathfrak{N}$ of K such that, for any λ, the Artin conductor of $\bar\rho_\lambda$ divides $\mathfrak{N}$.

5. Locally algebraic representations

We recall Serre's theory of locally algebraic Galois representations [**Se3**] (see also [**Ri**]).

Suppose that E is an algebraic number field (of either finite or infinite degree over $\mathbb{Q}$). We denote by E_λ the completion of E at a finite place λ of E. Let S be a finite set of finite places of K and T a finite set of finite places of E. Put $S_\ell := S \cup \{\text{places of } K \text{ above } \ell\}$. Let $\ell = \ell_\lambda$ be the prime number under a finite place λ of E. Let $T = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ be the torus over $\mathbb{Q}$ obtained from the multiplicative group $\mathbb{G}_m$ over K by restriction of scalars to $\mathbb{Q}$. We write $T_{/E_\lambda}$ for the base change $T \otimes_{\mathbb{Q}} E_\lambda$. Let $K_\ell := K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \prod_{v|\ell} K_v$, and regard its multiplicative group as a subgroup of the idèle group I_K of K. Note that $K_\ell^\times = T_{/\mathbb{Q}_\ell}(\mathbb{Q}_\ell) \subset T_{/E_\lambda}(E_\lambda)$. An abelian representation $\rho\colon G_K^{\mathrm{ab}} \to GL_n(E_\lambda)$ may be thought of as a representation of I_K by means of class field theory. Now ρ is said to be *locally algebraic* if there exists a morphism $r\colon T_{/E_\lambda} \to GL_{n/E_\lambda}$ of algebraic groups defined over E_λ such that

$$\rho(x) = r(x^{-1})$$

for all $x \in K_\ell^\times$ close enough to 1. For any finite extension L of K, ρ is locally algebraic if and only if $\rho|_{G_L}$ is locally algebraic. For a potentially abelian representation $\rho\colon G_K \to GL_n(E_\lambda)$, we say that ρ is locally algebraic if $\rho|_{G_L}$ is locally algebraic where L is a finite extension of K such that $\rho|_{G_L}$ is abelian.

Let $\mathfrak{m} = (m_v)_{v \in S(\mathfrak{m})}$ be a modulus of K, that is, a family of positive integers m_v for v in a finite set $S(\mathfrak{m})$ of finite places of K (we set $m_v := 0$ if $v \notin S(\mathfrak{m})$). We say that $\mathfrak{m}$ is a *modulus of definition for an abelian locally algebraic representation* ρ if:

(1) ρ is trivial on $U_{v,\mathfrak{m}}$ for each $v \in S(\mathfrak{m})$ with $v \nmid \ell$; and
(2) $\rho(x) = r(x^{-1})$ for $x \in U_{\ell,\mathfrak{m}}$.

Here, $U_{v,\mathfrak{m}}$ is the group of units u of $K_v^\times$ such that $u \equiv 1$ modulo the m_v-th power of a uniformizer of $K_v^\times$ if v is a finite place (resp. it is the connected component of $K_v^\times$ if v is an infinite place), $U_{\ell,\mathfrak{m}} = \prod_{v|\ell} U_{v,\mathfrak{m}}$, and $r\colon T_{/E_\lambda} \to GL_{n/E_\lambda}$ is as above. An abelian representation ρ is locally algebraic if and only if ρ is geometric in the sense of Fontaine-Mazur ([**FM**], Section 6, Proposition). More precisely, an abelian representation ρ is locally algebraic with a modulus $\mathfrak{m}$ if and only if it becomes semistable when restricted to $U_{\ell,\mathfrak{m}}$.

Put $U_\mathfrak{m} := \prod_v U_{v,\mathfrak{m}}$ (v runs through all the places of K), $I_\mathfrak{m} := I_K/U_\mathfrak{m}$ and $T_\mathfrak{m}$ the quotient of T by the Zariski closure of $K^\times \cap U_\mathfrak{m}$. Let $C_\mathfrak{m} := I_K/(K^\times U_\mathfrak{m})$ be the ray class group of K of modulus $\mathfrak{m}$. Then there exists a commutative algebraic group $S_\mathfrak{m}$ over $\mathbb{Q}$ which satisfies the following properties:

there exist an exact sequence

$$1 \to T_\mathfrak{m} \to S_\mathfrak{m} \to C_\mathfrak{m} \to 1$$

and a group homomorphism $\varepsilon\colon I_\mathfrak{m} \to S_\mathfrak{m}(\mathbb{Q})$ which make a following diagram commutative:

$$\begin{tikzcd}
1 \arrow[r] & K^\times/(K^\times \cap U_\mathrm{m}) \arrow[r] \arrow[d] & I_\mathfrak{m} \arrow[r] \arrow[d, "\varepsilon"] & C_\mathfrak{m} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & T_\mathfrak{m}(\mathbb{Q}) \arrow[r] & S_\mathfrak{m}(\mathbb{Q}) \arrow[r] & C_\mathfrak{m} \arrow[r] & 1.
\end{tikzcd}$$

An algebraic homomorphism $\phi\colon S_\mathfrak{m} \to GL_n$ over E induces a λ-adic representation $\phi_\lambda\colon G_K \to GL_n(E_\lambda)$ as follows: Denote by α_λ the composite map

$$I_K \to T(\mathbb{Q}_\ell) \to T_\mathfrak{m}(\mathbb{Q}_\ell) \hookrightarrow S_\mathfrak{m}(\mathbb{Q}_\ell) \hookrightarrow S_\mathfrak{m}(E_\lambda),$$

where the first arrow is the projection of I_K to its ℓ-th factor $K_\ell^\times = T(\mathbb{Q}_\ell)$ and the latter three arrows are canonical maps. Regarding ε as a map $I_K \to S_\mathfrak{m}(E_\lambda)$, it is not difficult to check that $\varepsilon = \alpha_\lambda$ on $K^\times$. Therefore, putting $\varepsilon_\lambda := \varepsilon\dot{\alpha}_\lambda^{-1}$, we see that $\varepsilon_\lambda\colon I_K \to S_\mathfrak{m}(E_\lambda)$ factors through $I_K/K^\times$ and, by class field theory, ε_λ defines a map $\varepsilon_\lambda\colon G_K^{\mathrm{ab}} \to S_\mathfrak{m}(E_\lambda)$. Now we denote by ϕ_λ the composite map

$$G_K^{\mathrm{ab}} \xrightarrow{\varepsilon_\lambda} S_\mathfrak{m}(E_\lambda) \xrightarrow{\phi} GL_n(E_\lambda).$$

By construction, ϕ_λ is abelian, semisimple, locally algebraic and E-rational with ramification set $S(\mathfrak{m})$. Moreover, $(\phi_\lambda)_\lambda$ is an E-rational strict compatible system of abelian semisimple λ-adic representations with empty defect set and ramification set $S(\mathfrak{m})$ ([**Se4**], Section II, Section 2.5, Theorem or, [**Ri**], Proposition (1.4.4)).

THEOREM 1.16 ([**He**], Théorèm 2). *Let E be a finite extension of $\mathbb{Q}$ and λ a finite place of E. Then any E-rational abelian semisimple λ-adic representation $\rho_\lambda\colon G_K \to GL_n(E_\lambda)$ is locally algebraic.*

THEOREM 1.17 ([**Ri**], Theorem (MT 1)). *Let E be an algebraic extension of $\mathbb{Q}$ and λ a finite place of E. Then any E-rational abelian semisimple locally algebraic λ-adic representation $\rho_\lambda \colon G_K \to GL_n(E_\lambda)$ with a modulus of definition $\mathfrak{m}$ arises from a unique semisimple representation $\phi \colon S_\mathfrak{m} \to GL_n$ over E.*

6. Hecke characters

In this section, we recall the construction of a Galois representation arising from a Hecke character. We denote by $(K_\infty^\times)^0$ the connected component of the identity of the product of the completions of K at the archimedean places and c complex conjugation. For any $z \in \mathbb{C}$, denote by $\bar{z}$ the complex conjugation $c(z)$ of z.

DEFINITION 1.18. A *Hecke character* is a continuous homomorphism $\psi \colon I_K/K^\times \to \mathbb{C}^\times$ such that

$$\psi|_{(K_\infty^\times)^0}(x) = \prod_{\sigma\ real} x_\sigma^{n_\sigma} \prod_{\sigma\ complex} x_\sigma^{n_\sigma} \bar{x}_{c\sigma}^{n_{c\sigma}} \quad (*)$$

for integers $n_\sigma, n_{c\sigma}$ and with x_σ the components of x. We say that the tuple of integers $(n_\sigma)_\sigma$ is the infinity type of ψ, and say that ψ is unramified at a finite place v if the units U_v at v are in the kernel of ψ. The conductor of ψ is the largest ideal $\mathfrak{N}$ such that elements of the finite ideles $I^{(\infty)}$ congruent to 1 mod $\mathfrak{N}$ are in the kernel of ψ.

Now we construct a representation ψ_λ from a Hecke character ψ. We want to use class field theory: $G_K^{\mathrm{ab}} \simeq I_K/\overline{K^\times(K_\infty^\times)^0}$, where $\overline{K^\times(K_\infty^\times)^0}$ is the topological closure of $K^\times(K_\infty^\times)^0$.

Let ℓ be a prime number. Let $\psi_0 \colon I_K \to \mathbb{C}^\times$ be a homomorphism defined by

$$\psi_0(x) = \psi(x) \prod_{\sigma\ real} x_\sigma^{-n_\sigma} \prod_{\sigma\ complex} x_\sigma^{-n_\sigma} \bar{x}_{c\sigma}^{-n_{c\sigma}},$$

then its kernel is open and takes values in a sufficiently large subfield E of $\mathbb{C}$ which is a finite extension of $\mathbb{Q}$[5], and thus we may regard ψ_0 as a continuous representation $I_K \to E^\times$. By definition, this character ψ_0 factors through the quotient $I_K/(K_\infty^\times)^0$, however, ψ_0 is not trivial on $K^\times$. Thus we modify ψ_0 by changing the image of the ℓ-part $K_\ell^\times := (K \otimes_\mathbb{Q} \mathbb{Q}_\ell)^\times \simeq \prod_{v|\ell} K_v^\times$ of I_K. Suppose that E contains the Galois closure of K and take any finite place λ of E above ℓ. Let $\eta \colon K^\times \to$

[5]This can be checked as follows: Since ψ_0 is continuous, there exists an open subgroup U of I_K such that ψ_0 factors through I_K/U. It is known that the image of I_K/U in the idele class group $I_K/K^\times$ of K is finite. Denote by X a set of representative of $I_K/UK^\times$, which is a finite set. If $x \in U \cdot K^\times$, then $\psi_0(x)$ has values in the Galois closure K^{Gal} of K by definition of a Hecke character. On the other hand, since X is finite, there exists an integer t such that, for any $x \in X$, x^t is in $U \cdot K^\times$ and thus $\psi_0(x)^t$ has values in K^{Gal}. Thus $\psi_0(X)$ is contained in a finite extension E of K^{Gal}. Therefore, the image of ψ_0 is contained in E.

$E_\lambda^\times$ be the homomorphism defined by $\eta(x) = \prod_\sigma \sigma(x)^{n_\sigma}$ and extend η to $\eta_\ell\colon (K \otimes \mathbb{Q}_\ell)^\times \to E_\lambda^\times$. From this η_ℓ, we obtain a continuous homomorphism $\psi_\lambda := \psi_0 \cdot (\eta_\ell \circ \alpha_\ell)$, where $\alpha_\ell\colon I_K \to (K \otimes \mathbb{Q}_\ell)^\times$ is the projection. Using the isomorphism of class field theory $G_K^{\mathrm{ab}} \simeq I_K/\overline{K^\times (K_\infty^\times)^0}$, we obtain a continuous character $\psi_\lambda\colon G_K \to E_\lambda^\times$.

Since ψ_λ is continuous, we know that ψ_λ has values in the group of units of E_λ and thus we obtain a mod λ representation $\bar{\psi}_\lambda\colon G_K \to \mathbb{F}_\lambda^\times$ where $\mathbb{F}_\lambda$ is the residue field of λ.

Denote by π_v a uniformizer of K_v for any finite place v.

Proposition 1.19. *Let the notation as above.*
(1) *ψ_λ is unramified away from the conductor $\mathfrak{N}$ of ψ. Moreover, ψ_λ is trivial on $\ker(U_v \to (\mathcal{O}_v/\mathfrak{n}\mathcal{O}_v)^\times)$ for any place v of K away from ℓ.*
(2) *For any finite place v away from $\mathfrak{N}$ and ℓ,*

$$\psi_\lambda(\mathrm{Fr}_v) = \psi_\lambda(\pi_v) = \psi(\pi_v) = \psi_0(\pi_v) \in E.$$

In particular, $\psi_\lambda(\mathrm{Fr}_v)$ is independent of the choice of λ and has values in E.
(3) *The system $(\psi_\lambda)_\lambda$ forms an E-rational strictly compatible system of 1-dimensional λ-adic representations of G_K with bounded conductor and bounded Hodge-Tate weights.*
(4) *The system $(\bar{\psi}_\lambda)_\lambda$ forms an E-rational strictly compatible system of 1-dimensional mod λ representations of G_K with bounded Artin conductor. In fact Artin conductor of $\bar{\psi}_\lambda$ is bounded by the conductor of ψ.*

Proof. The assertions (1) and (2) follows directly from the construction of a representation arising from a Hecke character. The boundedness of the conductor The assertion (3) follows from the existence of the conductor of a Hecke character and the boundedness of Hodge-Tate weights follows from the equation $(*)$ in Definition 1.18. The assertion (4) follows easily from the definition of the conductor of a Hecke character and Example 1.12. $\square$

In the case of 1-dimensional representations, Proposition 1.11 is written as follows.

Proposition 1.20. *Let $\psi\colon G_K \to E_\lambda^\times$ be a 1-dimensional λ-adic representation. Let ℓ be a residual characteristic of λ. Then the following are equivalent:*
(1) *ψ is locally algebraic.*
(2) *ψ arises from a Hecke character.*
(3) *ψ is Hodge-Tate at all finite places v above ℓ.*
(4) *ψ is de Rham at all finite places v above ℓ.*
(5) *ψ is potentially semistable at all finite places v above ℓ.*
(6) *ψ is potentially crystalline at all finite places v above ℓ.*
(7) *ψ comes from CM abelian varieties (up to a finite image).*

Khare proved in [**Kh2**] that every E-rational strictly compatible system of abelian representations arises motivically.

THEOREM 1.21 ([**Kh2**], Theorem 2 and Corollary 1). *An E-rational strictly compatible system of abelian semisimple λ-adic (resp. mod λ) representations of G_K arises from n Hecke characters.*

COROLLARY 1.22. (1) *An E-rational strictly compatible system $(\rho_\lambda)_\lambda$ of abelian semisimple λ-adic representations of G_K has bounded inertial level and bounded Hodge-Tate weights.*
(2) *An E-rational strictly compatible system $(\rho_\lambda)_\lambda$ of abelian semisimple mod λ representations of G_K is of bounded Artin conductor.*

PROOF. By Theorem 1.21, such $(\rho_\lambda)_\lambda$ arises from Hecke characters. Hence the Proposition follows from standard properties of a representation arising from Hecke characters (cf. Proposition 1.19). $\square$

CHAPTER 2

Tame inertia weights

In this chapter, we recall the definition of the tame inertia weights (cf. [**Se4**], Section 1) and Caruso's work on the tame inertia weights of a residual representation of a semistable Galois representation (cf. [**Ca1**]). Furthermore, in the last section, we consider the relationship between classical polygons (that is, the Hodge polygon and the Newton polygon introduced in [**Fo1**]) and the tame inertia polygon (which is proposed in [**CS**]).

In this chapter, we write K for a complete discrete valuation field of characteristic zero with perfect residue field k of positive characteristic p.

1. Definition of tame inertia weights

We denote by I the inertia subgroup of G_K, I_w its wild inertia subgroup and $I_t := I/I_w$ the tame inertia group. Let V be an h-dimensional irreducible $\mathbb{F}_p$-representation of I and fix an algebraic closure $\bar{k}$ of k. We denote by $\bar{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$ in $\bar{k}$ and $\mathbb{F}_{p^h}$ the finite subfield of $\bar{\mathbb{F}}_p$ with p^h-elements. Since V is irreducible and I_w is a normal subgroup of I, the action of I on V factors through I_t and thus we can regard[1] V as a representation of I_t. Applying Schur's lemma, we see that $\mathbb{E} := \mathrm{End}_{I_t}(V)$ is the finite field of order p^h. Moreover, the representation V inherits a structure of a 1-dimensional $\mathbb{E}$-representation of I_t by a natural manner. This representation is given by a character $\rho\colon I_t \to \mathbb{E}^\times$. Choose any isomorphism $f\colon \mathbb{E} \xrightarrow{\simeq} \mathbb{F}_{p^h}$ and consider the composition $\rho_f\colon I_t \xrightarrow{\rho} \mathbb{E}^\times \xrightarrow{f} \mathbb{F}_{p^h}^\times$:

$$\begin{tikzcd} I_t \arrow[r, "\rho"'] \arrow[rr, bend left, "\rho_f"] \arrow[dr] & \mathbb{E}^\times \arrow[r, "\simeq", "f"'] \arrow[d, hook] & \mathbb{F}_{p^h}^\times. \\ & GL(V) = \mathrm{End}_{\mathbb{F}_p}(V)^\times & \end{tikzcd}$$

Denote by $\mu_{p^h-1}(\bar{K})$ the set of (p^h-1)-st roots of unity in an algebraic closure $\bar{K}$ of K. Consider the isomorphism $\mu_{p^h-1}(\bar{K}) \simeq \mathbb{F}_{p^h}^\times$

[1] Since I_w is normal in I, we see that V^{I_w} is stable under the action of I. By the irreducibility of V, we have that V^{I_w} is 0 or V itself. Since I_w is pro-p, $V^{I_w} = V$ and this implies I_w acts on V trivially.

coming from the reduction map $\mathcal{O}_{\bar{K}} \twoheadrightarrow \bar{k}$, where $\mathcal{O}_{\bar{K}}$ is the integer ring of $\bar{K}$. Then a fundamental character of level h is given as follows:

$$\theta_h \colon I_t \to \mu_{p^h-1}(\bar{K}) \simeq \mathbb{F}_{p^h}^{\times}.$$
$$\sigma \mapsto \frac{\eta^{\sigma}}{\eta}$$

Here η is a $(p^h - 1)$-st root of a uniformizer of K. It is easy to check that $\theta_h^{1+p+\cdots+p^{h-1}} = \theta_1$, $\theta_h^{p^h-1} = 1$ and, with respect to h embeddings $\mathbb{F}_{p^h} \hookrightarrow \bar{\mathbb{F}}_p$, all the fundamental characters are given by $\theta_{h,0}(:= \theta_h), \theta_{h,1}, \theta_{h,2}, \ldots, \theta_{h,h-1}$, where $\theta_{h,i} = \theta_{h,i-1}^p$ for $0 \le i \le h-1$ and $\theta_{h,0} = \theta_{h,h-1}^p$. It is known that θ_1^e coincides with the mod p cyclotomic character ([**Se4**], Section 1.8, Proposition 8). Since I_t is pro-cyclic and $\mathrm{Im}(\theta_h) = \mathbb{F}_{p^h}^{\times}$, there exists an integer $n_f \in \{0, 1, \ldots, p^h - 2\}$ such that $\rho_f = \theta_h^{n_f}$. If we decompose $n_f = n_0 + n_1 p + n_2 p^2 + \cdots + n_{h-1} p^{h-1}$ with integers $0 \le n_i \le p - 1$ for any i, then we can see that the set $\{n_0, n_1, n_2, \ldots, n_{h-1}\}$ is independent of the choice of f.

Definition 2.1. We call these numbers $n_0, n_1, n_2, \ldots, n_{h-1}$ the *tame inertia weights of* V. In general, for any $\mathbb{F}_p$-representation V of I, the tame inertia weights of V are the tame inertia weights of all the Jordan-Hölder quotients of V.

Example 2.2. Suppose that k is algebraically closed. Let E be an elliptic curve over K with semistable reduction. If E has supersingular reduction, assume $e = 1$. Then the tame inertia weights of $E[p]$ are 0 and e (cf. [**Se4**], Section 1, Proposition 11 and 12).

Definition 2.3. Let V be a p-adic representation of G_K. The tame inertia weights of V are the tame inertia weights of a residual representation of $V|_I$.

The above definition is independent of the choice of a residual representation of V by the Brauer-Nesbitt theorem.

Definition 2.4. Let w be an integer with $0 \le w < p-1$ and V be an n-dimensional p-adic representation of G_K. Denote by $w_1 \le w_2 \le \cdots \le w_n$ all the tame inertia weights of V. We say that V is *of uniform tame inertia weight* w if $w_1 = w_2 = \cdots = w_n = w$.

2. Filtered (φ, N)-modules

We recall the theory of classical filtered (φ, N)-modules, which classify semistable p-adic representations of G_K.

Let $W(k)$ be the ring of Witt vectors of k and φ the Frobenius automorphism on k and $W(k)$. Put $K_0 := \mathrm{Fr}(W(k))$.

Definition 2.5. A *filtered* (φ, N)*-module* is a finite dimensional K_0-vector space D endowed with

- a Frobenius semilinear injection $\varphi\colon D \to D$.
- a K_0-linear map $N\colon D \to D$ such that $N\varphi = p\varphi N$.
- a decreasing filtration $(\mathrm{Fil}^i D_K)_{i\in\mathbb{Z}}$ on $D_K := D \otimes_{K_0} K$ by K-vector spaces such that $\mathrm{Fil}^i D_K = D_K$ for $i \ll 0$ and $\mathrm{Fil}^i D_K = 0$ for $i \gg 0$.

We denote by $\mathrm{MF}(\varphi, N)$ the category of filtered (φ, N)-modules[2]. The morphisms in $\mathrm{MF}(\varphi, N)$ are K_0-linear maps that preserve a filtration and commute with φ and N.

Let D be a filtered (φ, N)-module. Put $t_H(D) := \sum_{i\in\mathbb{Z}} i\cdot\dim_{K_0}\mathrm{gr}^i D_K$, where $\mathrm{gr}^i D_K := \mathrm{Fil}^i D_K/\mathrm{Fil}^{i+1} D_K$. For a rational number α, denote by D_α the slope α part of D associated with the action of the Frobenius of D. If k is algebraically closed and if $\alpha = r/s$ with $r, s \in \mathbb{Z}, s \geq 1$, then D_α is the sub K_0-vector space generated by $d \in D$ such that $\varphi^s(d) = p^r d$. We can check $D = \oplus_{\alpha\in\mathbb{Q}} D_\alpha$. It can be seen $\alpha d_\alpha \in \mathbb{Z}$, where $d_\alpha := \dim_{K_0} D_\alpha$. We put $t_N(D) := \sum_{\alpha\in\mathbb{Q}} \alpha \cdot \dim_{K_0} D_\alpha$.

Definition 2.6. A filtered (φ, N)-module D is called *weakly admissible* if $t_H(D) = t_N(D)$ and for any filtered (φ, N)-submodule $D' \subset D$, $t_H(D') \leq t_N(D')$.

We denote by $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$ the full subcategory of $\mathrm{MF}(\varphi, N)$ whose objects are weakly admissible filtered (φ, N)-modules.

Let V be a p-adic representation of G_K. Put $D_{\mathrm{st}}(V) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ and $D_{\mathrm{st}}(V)_K := D_{\mathrm{st}}(V) \otimes_{K_0} K$, where B_{st} is Fontaine's p-adic period ring (cf. [**Fo3**]). Then $D_{\mathrm{st}}(V)$ is a finite dimensional K_0 vector space and $\dim_{K_0} D_{st}(V) \leq \dim_{\mathbb{Q}_p} V$. We say that V is *semistable* if $\dim_{K_0} D_{\mathrm{st}}(V) = \dim_{\mathbb{Q}_p} V$. We equip $D_{\mathrm{st}}(V)$ with a structure of a filtered (φ, N)-module by a natural manner, that is, $\varphi_{D_{\mathrm{st}}(V)} := \varphi_{B_{\mathrm{st}}} \otimes 1, N_{D_{\mathrm{st}}(V)} := N_{B_{\mathrm{st}}} \otimes 1$, and $\mathrm{Fil}^i(D_{\mathrm{st}}(V)_K) := (\mathrm{Fil}^i B_{\mathrm{st}} \otimes V)^{G_K} \otimes_{K_0} K$. The aforementioned result of Colmez and Fontaine is

Theorem 2.7 ([**CF**]). *The functor $D_{\mathrm{st}}\colon V \to (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ establishes an equivalence of categories between the category of semistable p-adic representations of G_K and the category of weakly admissible filtered (φ, N)-modules. A quasi-inverse V_{st} to D_{st} is given by $V_{\mathrm{st}}(D) := \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_K) \cap (B_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=1, N=0}$.*

We note that, if we put $D^*_{\mathrm{st}}(V) := D_{\mathrm{st}}(V^\vee)$, then the Hodge-Tate weights of V is exactly the $i \in \mathbb{Z}$ such that $\mathrm{gr}^i D^*_{\mathrm{st}}(V)_K \neq 0$. A quasi-inverse V^*_{st} to D^*_{st} is given by $V^*_{\mathrm{st}}(D) := \mathrm{Hom}_{\mathrm{Fil}^\cdot, \varphi, N}(D, B_{\mathrm{st}})$.

3. Breuil modules

From now on, we fix a uniformizer π in K and denote by $E(u)$ its minimal polynomial over K_0 (which is an Eisenstein polynomial of

[2]Of course, this category $\mathrm{MF}(\varphi, N)$ depends on the field K. Thus it should be denoted by $\mathrm{MF}_K(\varphi, N)$. But for simplicity, we use notation $\mathrm{MF}(\varphi, N)$ instead of $\mathrm{MF}_K(\varphi, N)$

degree $e := [K : K_0]$). Let S be the p-adic completion of $W(k)[u, \frac{E(u)^i}{i!}]$ where u is an indeterminate and endow S with the following structures:

- a continuous φ-semilinear Frobenius $\varphi\colon S \to S$ defined by $\varphi(u) := u^p$.
- a continuous linear derivation $N\colon S \to S$ defined by $N(u) = -u$.
- a decreasing filtration $(\mathrm{Fil}^i S)_{i\in\mathbb{Z}_{\geq 0}}$ where $\mathrm{Fil}^i S$ is the p-adic completion of $\sum_{j\geq i} S\frac{E(u)^j}{j!}$.

We can check that $N\varphi = p\varphi N$, $N(\mathrm{Fil}^{i+1}S) \subset \mathrm{Fil}^i S$ $(i \in \mathbb{Z}_{\geq 0})$, $\varphi(\mathrm{Fil}^i S) \subset p^i S$ $(0 \leq i \leq p-1)$. In particular, $\varphi_i := \frac{1}{p^i}\varphi\colon \mathrm{Fil}^i S \to S$ is well-defined for any $0 \leq i \leq p-1$. Putting $c := \varphi_1(E(u))$, we see $c \in S^\times$.

Put $S_{K_0} := S \otimes_{W(k)} K_0 = S[1/p]$. We extend φ and N of S to S_{K_0}, and write $\mathrm{Fil}^i S_{K_0} := \mathrm{Fil}^i S \otimes_{W(k)} K_0 = (\mathrm{Fil}^i S)[1/p]$ for all $i \geq 0$.

Definition 2.8. We define $\mathcal{MF}(\varphi, N)$ to be the category whose objects are finite free S_{K_0}-modules $\mathcal{D}$ endowed with

- a $\varphi_{S_{K_0}}$-semilinear homomorphism $\varphi\colon \mathcal{D} \to \mathcal{D}$ such that the determinant of φ is invertible in S_{K_0}.
- a decreasing filtration $(\mathrm{Fil}^i\mathcal{D})_{i\in\mathbb{Z}}$ such that $\mathrm{Fil}^0\mathcal{D} = 0$ and that $\mathrm{Fil}^i S_{K_0} \cdot \mathrm{Fil}^j\mathcal{D} \subset \mathrm{Fil}^{i+j}\mathcal{D}$.
- a K_0-linear endomorphism $N\colon \mathcal{D} \to \mathcal{D}$ which satisfy the following:
  (1) $N(sx) = N(s)x + sN(x)$ for $s \in S_{K_0}$ and $x \in \mathcal{D}$,
  (2) $N\varphi = p\varphi N$,
  (3) $N(\mathrm{Fil}^i\mathcal{D}) \subset \mathrm{Fil}^{i-1}\mathcal{D}$.

The morphisms in $\mathcal{MF}(\varphi, N)$ are S-linear maps that preserve filtration and commute with φ and N. The above φ and N on $\mathcal{D}$ are called *Frobenius* and *monodromy operator*, respectively. An object in this category $\mathcal{MF}(\varphi, N)$ is called a *Breuil module*.

Let D be a filtered (φ, N)-module. We can associate an object $\mathcal{D} = \mathcal{D}(D) \in \mathcal{MF}(\varphi, N)$ by the following:

$$\mathcal{D} := S \otimes_{W(k)} D$$

and

- $\varphi := \varphi_S \otimes \varphi_D\colon \mathcal{D} \to \mathcal{D}$.
- $N := N_S \otimes \mathrm{Id} + \mathrm{Id} \otimes N_D$.
- $\mathrm{Fil}^0\mathcal{D} := \mathcal{D}$ and by induction;

  $$\mathrm{Fil}^{i+1}\mathcal{D} := \{x \in \mathcal{D} \mid N(x) \in \mathrm{Fil}^i\mathcal{D},\ f_\pi(x) \in \mathrm{Fil}^{i+1}D_K\}$$

  where $f_\pi\colon \mathcal{D} \to D_K$ is defined by $s(u) \otimes x \mapsto s(\pi)x$.

Let $R := \varprojlim \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ where the transition maps are given by the p-th power map. We fix a sequence $(\pi_n)_{n\geq 0}$ of elements $\pi_n \in \bar{K}$ such that $\pi_0 := \pi$ and $\pi_{n+1}^p = \pi_n$ for any non-negative integer n. Write

$\underline{\pi}$:= (π_n mod p) $\in R$, and let $[\underline{\pi}] \in W(R)$ be the Teichmüller lift. Let $\hat{A}_{\mathrm{st}}$ be the p-adic completion of $A_{\mathrm{cris}}\langle X\rangle = A_{\mathrm{cris}}[X, \frac{X^i}{i!}]_{i\geq 0}$; explicitly,

$$\hat{A}_{\mathrm{st}} = \{\sum_{i\geq 0} a_i \frac{X^i}{i!} \mid a_i \in A_{\mathrm{cris}}, a_i \to 0\}.$$

We endow $\hat{A}_{\mathrm{st}}$ with the following structures:

- a continuous φ-semilinear Frobenius $\varphi\colon \hat{A}_{\mathrm{st}} \to \hat{A}_{\mathrm{st}}$ defined by $\varphi(X) := (1+X)^p - 1$.
- a continuous A_{cris}-linear homomorphism $N\colon \hat{A}_{\mathrm{st}} \to \hat{A}_{\mathrm{st}}$ defined by $N(X) := 1 + X$.
- a decreasing filtration
$$\mathrm{Fil}^n \hat{A}_{\mathrm{st}} = \{\sum_{i\geq 0} a_i \frac{X^i}{i!} \mid a_i \in \mathrm{Fil}^{n-i} A_{\mathrm{cris}}, a_i \to 0\},$$
where $\mathrm{Fil}^k A_{\mathrm{cris}} := A_{\mathrm{cris}}$ for $k \leq 0$.
- a G_K-action on $\hat{A}_{\mathrm{st}}$ defined by $g.X := [\underline{\varepsilon}(g)]X + [\underline{\varepsilon}(g)] - 1$, where $\underline{\varepsilon}(g) := g(\underline{\pi})/\ \underline{\pi} \in R$.

We can check that $N\varphi = p\varphi N$, $N(\mathrm{Fil}^{i+1}\hat{A}_{\mathrm{st}}) \subset \mathrm{Fil}^i \hat{A}_{\mathrm{st}}$ ($i \in \mathbb{Z}_{\geq 0}$) and $\varphi(\mathrm{Fil}^i \hat{A}_{\mathrm{st}}) \subset p^i \hat{A}_{\mathrm{st}}$ ($0 \leq i \leq p-1$). In particular, $\varphi_i := \frac{1}{p^i}\varphi\colon \mathrm{Fil}^i \hat{A}_{\mathrm{st}} \to \hat{A}_{\mathrm{st}}$ is well-defined for any $0 \leq i \leq p-1$. Furthermore, a G_K-action on $\hat{A}_{\mathrm{st}}$ preserves the filtration and commutes with φ and N. The homomorphism[3] $S \to \hat{A}_{\mathrm{st}}$, defined by $u \mapsto [\underline{\pi}]/(1+X)$, induces an isomorphism $S \xrightarrow{\simeq} (\hat{A}_{\mathrm{st}})^{G_K}$ (cf. [**Br1**], Proposition 4.1.2), and by this isomorphism we regard $\hat{A}_{\mathrm{st}}$ as an S-algebra.

For any $\mathcal{D} \in \mathcal{MF}(\varphi, N)$, one can associate a $\mathbb{Q}_p[G_K]$-module

$$V^*_{\mathrm{st}}(\mathcal{D}) := \mathrm{Hom}_{S,\mathrm{Fil}^\cdot,\varphi,N}(\mathcal{D}, \hat{A}_{\mathrm{st}}[1/p]).$$

THEOREM 2.9 ([**Br1**]). *The functor $\mathcal{D}\colon D \mapsto S \otimes_{W(k)} D (:= \mathcal{D})$ induces an equivalence between the categories $\mathcal{MF}(\varphi, N)$ and* $\mathrm{MF}(\varphi, N)$ *and there exists a natural isomorphism*[4] $V^*_{\mathrm{st}}(\mathcal{D}) \simeq V^*_{\mathrm{st}}(D)$.

We denote by $\mathcal{MF}^{\mathrm{w}}(\varphi, N)$ the essential image of $\mathcal{D}$ restricted to $\mathrm{MF}^{\mathrm{w}}(\varphi, N)$.

$$\begin{array}{ccc} \mathrm{MF}(\varphi, N) & \xrightarrow{\mathcal{D}} & \mathcal{MF}(\varphi, N) \\ \cup & & \cup \\ \mathrm{MF}^{\mathrm{w}}(\varphi, N) & \xrightarrow{\mathcal{D}} & \mathcal{MF}^{\mathrm{w}}(\varphi, N). \end{array}$$

[3]By definition, this morphism preserves the filtration and commutes with φ and N.

[4]In particular, $\dim_{\mathbb{Q}_p} V^*_{\mathrm{st}}(\mathcal{D}) = \dim_{\mathbb{Q}_p} V^*_{\mathrm{st}}(D) = \dim_{K_0} D = \mathrm{rank}_{S_{K_0}} \mathcal{D}$.

4. Integral structures

From now on, we fix an integer $r \geq 0$ such that $r < p-1$.

Definition 2.10. Let D be an weakly admissible filtered (φ, N)-module satisfying $\mathrm{Fil}^0 D_K = 0$ and $\mathrm{Fil}^{r+1} D_K = 0$. Let $\mathcal{D} := \mathcal{D}(D) \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$. A *strongly divisible lattice (or, strongly divisible module) of weight $\leq r$ in $\mathcal{D}$* is an S-submodule $\mathcal{M}$ of $\mathcal{D}$ such that

- $\mathcal{M}$ is finite free as an S-module and $\mathcal{M}[1/p] \simeq \mathcal{D}$.
- $\mathcal{M}$ is stable under $\varphi_{\mathcal{D}}$.
- $\varphi_{\mathcal{D}}(\mathrm{Fil}^r \mathcal{M}) \subset p^r \mathcal{M}$ where $\mathrm{Fil}^r \mathcal{M} := \mathcal{M} \cap \mathrm{Fil}^r \mathcal{D}$.
- $\mathcal{M}$ is stable under $N_{\mathcal{D}}$.

A *strongly divisible lattice of weight* $\leq r$ is a strongly divisible lattice in some $\mathcal{D}$ as above. A morphism between two strongly divisible lattices of weight $\leq r$ is an S-linear morphism which commutes the additional structures.

Definition 2.11. The category $'\mathrm{Mod}_{/S}^{r,\varphi,N}$ is the category whose objects are S-modules $\mathcal{M}$ endowed with an S-submodule $\mathrm{Fil}^r \mathcal{M}$ of $\mathcal{M}$, a φ_S-semilinear homomorphism $\varphi_r \colon \mathrm{Fil}^r \mathcal{M} \to \mathcal{M}$ and a $W(k)$-linear endomorphism $N \colon \mathcal{M} \to \mathcal{M}$ which satisfy the following:

- $\mathrm{Fil}^r S \cdot \mathcal{M} \subset \mathrm{Fil}^r \mathcal{M}$.
- $\varphi_r(sx) = \frac{1}{c^r} \varphi_r(s) \varphi_r(E(u)^r x)$ for $s \in \mathrm{Fil}^r S$ and $x \in \mathcal{M}$.
- $N(sx) = N(s)x + sN(x)$ for $s \in S$ and $x \in \mathcal{M}$.
- (*Griffiths transversality*) $E(u)N(\mathrm{Fil}^r \mathcal{M}) \subset \mathrm{Fil}^r \mathcal{M}$.
- The following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Fil}^r \mathcal{M} \arrow[r, "\varphi_r"] \arrow[d, "E(u)N"'] & \mathcal{M} \arrow[d, "cN"] \\
\mathrm{Fil}^r \mathcal{M} \arrow[r, "\varphi_r"] & \mathcal{M}.
\end{tikzcd}$$

The morphisms in $'\mathrm{Mod}_{/S}^{r,\varphi,N}$ are S-linear maps that preserve Fil^r and commute with φ_r and N. The above φ_r and N on $\mathcal{M}$ are called *Frobenius* and *monodromy operator*, respectively.

Definition 2.12. The category $\mathrm{ModFI}_{/S}^{r,\varphi,N}$ is the full subcategory of $'\mathrm{Mod}_{/S}^{r,\varphi,N}$ whose objects $\mathcal{M}$ satisfy the following:

- $\mathcal{M}$ is of finite length as a $\mathbb{Z}_p$-module.
- $\mathrm{Im}(\varphi_r)$ generates $\mathcal{M}$ as an S-modules.

Definition 2.13. The category $\mathrm{Mod}_{/S}^{r,\varphi,N}$ is the full subcategory of $'\mathrm{Mod}_{/S}^{r,\varphi,N}$ whose objects $\mathcal{M}$ satisfy the following:

- $\mathcal{M}$ is finite free as an S-module.
- $(\mathcal{M}_n, \mathrm{Fil}^r \mathcal{M}_n, \varphi_r, N) \in \mathrm{ModFI}_{/S}^{r,\varphi,N}$ for all positive integers n.

Here, $\mathcal{M}_n := \mathcal{M}/p^n \mathcal{M}, \mathrm{Fil}^r \mathcal{M}_n := \mathrm{Fil}^r \mathcal{M}/p^n \mathrm{Fil}^r \mathcal{M}$, and φ_r, N on $\mathcal{M}$ induce those of $\mathcal{M}_n$.

By definition, we can regard a strongly divisible module of weight $\leq r$ as an object in the category $\mathrm{Mod}^{r,\varphi,N}_{/S}$.

Note that $\hat{A}_{\mathrm{st}} \in {}'\mathrm{Mod}^{r,\varphi,N}_{/S}$. For any $\mathcal{M} \in \mathrm{Mod}^{r,\varphi,N}_{/S}$, define

$$T^*_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}_{'\mathrm{Mod}^{r,\varphi,N}_{/S}}(\mathcal{M}, \hat{A}_{\mathrm{st}})$$

and endow it with a G_K-action by $(g.f)(x) := g(f(x))$ for $g \in G_K$, $f \in T^*_{\mathrm{st}}(\mathcal{M})$ and $x \in \mathcal{M}$.

Proposition 2.14 ([**Br4**]). *The category of strongly divisible lattices of weight $\leq r$ is just $\mathrm{Mod}^{r,\varphi,N}_{/S}$.*

Let $\mathcal{M} \in \mathrm{Mod}^{r,\varphi,N}_{/S}$. By Proposition 2.14, there exists $\mathcal{D} \in \mathcal{MF}^{\mathrm{w}}(\varphi, N)$, which correspond to $D \in \mathrm{MF}(\varphi, N)$ satisfying $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^{r+1} D_K = 0$, such that $\mathcal{M}$ is a strongly divisible lattice in $\mathcal{D}$. We can check that

$$\begin{aligned} V^*_{\mathrm{st}}(D) \simeq V^*_{\mathrm{st}}(\mathcal{D}) &= \mathrm{Hom}_{S,\mathrm{Fil}^{\cdot},\varphi,N}(\mathcal{D}, \hat{A}_{\mathrm{st}}[1/p]) \\ &= \mathrm{Hom}_{S,\mathrm{Fil}^{r},\varphi,N}(\mathcal{D}, \hat{A}_{\mathrm{st}}[1/p]) \simeq T^*_{\mathrm{st}}(\mathcal{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \end{aligned}$$

as $\mathbb{Q}_p[G_K]$-modules[5]. On the other hand, since $\mathcal{M}$ is free over S, $T^*_{\mathrm{st}}(\mathcal{M})$ is a free $\mathbb{Z}_p$-module. Therefore, we know that $T^*_{\mathrm{st}}(\mathcal{M})$ is a G_K-stable $\mathbb{Z}_p$-lattice of $V^*_{\mathrm{st}}(D) \simeq V^*_{\mathrm{st}}(\mathcal{D})$ and $\mathrm{rank}_{\mathbb{Z}_p} T^*_{\mathrm{st}}(\mathcal{M}) = \mathrm{rank}_S \mathcal{M}$.

5. Simple objects in $\mathrm{ModFI}^{r,\varphi,N}_{/S}$

Suppose $er < p-1$. We will see later that a simple object in $\mathrm{ModFI}^{r,\varphi,N}_{/S}$ corresponds to a Jordan-Hölder quotient of a residual representation of a semistable Galois representation, and hence we want to know properties of simple objects in $\mathrm{ModFI}^{r,\varphi,N}_{/S}$. Since any simple object $\mathcal{M}$ in $\mathrm{ModFI}^{r,\varphi,N}_{/S}$ is killed[6] by p, we may restrict our attention to the full subcategory of $\mathrm{ModFI}^{r,\varphi,N}_{/S}$ whose objects are killed by p.

Let $\sigma\colon S/pS \to k[u]/u^{ep}$ be a surjective k-algebra homomorphism defined by $\sigma(u) := u$ and $\sigma(u^{ei}/i!) := 0$ for $i \geq p$. By this homomorphism, we regard $k[u]/u^{ep}$ as an S/pS-module. Put $\bar{c} := \sigma(c) \in$

[5]By Theorem 2.9, only a non-trivial part of this equation is $\mathrm{Hom}_{S,\mathrm{Fil}^{\cdot},\varphi,N}(\mathcal{D}, \hat{A}_{\mathrm{st}}[1/p]) = \mathrm{Hom}_{S,\mathrm{Fil}^{r},\varphi,N}(\mathcal{D}, \hat{A}_{\mathrm{st}}[1/p])$. This can be checked as follows: It is enough to show that, if f is in $\mathrm{Hom}_{S,\mathrm{Fil}^{r},\varphi,N}(\mathcal{D}, \hat{A}_{\mathrm{st}}[1/p])$, then f preserves Fil^i for $0 \leq i \leq r$. Let $0 \leq i \leq r$. Take any $x \in \mathrm{Fil}^{r-i}\mathcal{D}$. Since $\mathrm{Fil}^i S_{K_0} \cdot \mathrm{Fil}^{r-i}\mathcal{D} \subset \mathrm{Fil}^r \mathcal{D}$, we have $E([\underline{\pi}])^i f(x) = f(E(u)^i x) \in \mathrm{Fil}^r \mathcal{D}$. Writing $f(x) = \sum_{j=0}^{\infty} a_j \frac{X^j}{j!}$ where $a_j \in A_{\mathrm{cris}}$, we have $E([\underline{\pi}])^i a_j \in \mathrm{Fil}^{r-j} A_{\mathrm{cris}}[1/p]$. Hence have $a_j \in \mathrm{Fil}^{r-i-j} A_{\mathrm{cris}}[1/p]$ (cf. Lemma 3.2.2 of [**Li1**]). and therefore, we obtain $f(x) \in \mathrm{Fil}^{r-i} A_{\mathrm{cris}}[1/p]$. This shows that f preserves Fil^{r-i}.

[6]Suppose that there exists a simple object $\mathcal{M}$ with $p\mathcal{M} \neq 0$. Since the multiplication-by-p map $p|_{\mathcal{M}}$ on $\mathcal{M}$ is a morphism in $\mathrm{ModFI}^{r,\varphi,N}_{/S}$, $\ker(p|_{\mathcal{M}})$ is a non-trivial subobject of $\mathcal{M}$. This is a contradiction.

$(k[u]/u^{ep})^\times$. Note that $\sigma(E(u)) = u^e$. We endow $k[u]/u^{ep}$ with additional structures which are compatible with those of S via σ, explicitly,

- a continuous φ-semilinear Frobenius $\varphi\colon k[u]/u^{ep} \to k[u]/u^{ep}$ defined by $\varphi(u) := u^p$.
- a continuous linear derivation $N\colon k[u]/u^{ep} \to k[u]/u^{ep}$ defined by $N(u) = -u$.
- a decreasing filtration $(\mathrm{Fil}^i k[u]/u^{ep})_{i\in\mathbb{Z}_{\geq 0}}$ where $\mathrm{Fil}^i k[u]/u^{ep} := u^{ei}k[u]/u^{ep}$.

Definition 2.15. We define $\mathrm{ModFI}^{r,\varphi,N}_{/k}$ to be the category whose objects are $k[u]/u^{ep}$-modules $\mathcal{M}$ endowed with an $k[u]/u^{ep}$-submodule $\mathrm{Fil}^r\mathcal{M}$ of $\mathcal{M}$, a $\varphi_{k[u]/u^{ep}}$-semilinear homomorphism $\varphi_r\colon \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$ over $k[u]/u^{ep}$ and a k-linear endomorphism $N\colon \mathcal{M} \to \mathcal{M}$ which satisfy the following:

- $\mathrm{Fil}^r k[u]/u^{ep} \cdot \mathcal{M}(= u^{er}\mathcal{M}) \subset \mathrm{Fil}^r\mathcal{M}$.
- $\mathrm{Im}(\varphi_r)$ generates $\mathcal{M}$ as a $k[u]/u^{ep}$-modules.
- $N(\lambda x) = N(\lambda)x + \lambda N(x)$ for $\lambda \in k[u]/u^{ep}$ and $x \in \mathcal{M}$.
- $u^e N(\mathrm{Fil}^r\mathcal{M}) \subset \mathrm{Fil}^r\mathcal{M}$.
- The following diagram is commutative:

$$\begin{tikzcd} \mathrm{Fil}^r\mathcal{M} \arrow[r, "\varphi_r"] \arrow[d, "u^e N"] & \mathcal{M} \arrow[d, "\bar{c}N"] \\ \mathrm{Fil}^r\mathcal{M} \arrow[r, "\varphi_r"] & \mathcal{M}. \end{tikzcd}$$

It is known that the categories $\mathrm{ModFI}^{r,\varphi,N}_{/S}$ and $\mathrm{ModFI}^{r,\varphi,N}_{/k}$ are abelian and artinian category (cf. [**Ca1**], Section 3.5).

Let $\mathcal{M}$ be an object of $\mathrm{ModFI}^{r,\varphi,N}_{/S}$ killed by p. Regard $\mathcal{M}$ as an finite free S/pS-module and put $T(\mathcal{M}) := \mathcal{M} \otimes_{S/pS,\sigma} k[u]/u^{ep}$. By equipping $T(\mathcal{M})$ with natural φ_r, N and Fil^r arising from those of $\mathcal{M}$ and $k[u]/u^{ep}$, we see that $T(\mathcal{M})$ is an object of $\mathrm{ModFI}^{r,\varphi,N}_{/k}$.

Proposition 2.16 ([**Ca1**], Proposition 2.3.1). *The functor T induces an equivalence between the full subcategory of* $\mathrm{ModFI}^{r,\varphi,N}_{/S}$ *whose objects are killed by p and* $\mathrm{ModFI}^{r,\varphi,N}_{/k}$.

Proposition 2.17 ([**Ca1**], Proposition 3.2.1). *Let $\mathcal{M} \in \mathrm{ModFI}^{r,\varphi,N}_{/k}$ be free of rank d over $k[u]/u^{ep}$. Then there exist $k[u]/u^{ep}$-basis $e_1, e_2, \ldots, e_d$ of $\mathcal{M}$ and non-negative integers $n_1, n_2, \ldots, n_d$ such that*

$$\mathrm{Fil}^r\mathcal{M} = k[u]/u^{ep} \cdot u^{n_1}e_1 \oplus k[u]/u^{ep} \cdot u^{n_2}e_2 \oplus \cdots k[u]/u^{ep} \cdot u^{n_d}e_d.$$

Furthermore, the integers $n_1, n_2, \ldots, n_d$ are independent of the choice of basis.

We call such a basis $(e_1, e_2, \ldots, e_d)$ *adapted basis.* This proposition follows from the structure theorem of finitely generated modules over PID.

DEFINITION 2.18. Let $\mathfrak{n} = (n_i)_{i\in\mathbb{Z}/h\mathbb{Z}}$ be a sequence of integers $0 \le n_i \le er$ with period h. We define an object $\mathcal{M}(\mathfrak{n})$ of $\mathrm{ModFI}^{r,\varphi,N}_{/k}$ as follows:

- $\mathcal{M}(\mathfrak{n}) := \oplus_{i\in\mathbb{Z}/h\mathbb{Z}} k[u]/u^{ep} \cdot e_i$.
- $\mathrm{Fil}^r\mathcal{M}(\mathfrak{n}) := \oplus_{i\in\mathbb{Z}/h\mathbb{Z}} k[u]/u^{ep} \cdot u^{n_i} e_i$.
- $\varphi_r(u^{n_i}e_i) := e_{i+1}$ for $i \in \mathbb{Z}/h\mathbb{Z}$.
- $N(e_i) := 0$ for $i \in \mathbb{Z}/h\mathbb{Z}$.

PROPOSITION 2.19 ([**Ca1**], Théorème 4.3.2). *Suppose that k is algebraically closed. Then $\mathcal{M}(\mathfrak{n})$ is simple. Conversely, any simple object in $\mathrm{ModFI}^{r,\varphi,N}_{/k}$ is isomorphic to an object of the form $\mathcal{M}(\mathfrak{n})$.*

PROPOSITION 2.20 ([**Ca1**], Corollaire 4.3.4). *Let $\mathfrak{n} = (n_i)_{i\in\mathbb{Z}/h\mathbb{Z}}$ and $\mathfrak{m} = (m_i)_{i\in\mathbb{Z}/h'\mathbb{Z}}$ be sequences of period h and h', respectively. Then two objects $\mathcal{M}(\mathfrak{n})$ and $\mathcal{M}(\mathfrak{m})$ are isomorphic if and only if $h = h'$ and, for some integer a, $n_i = m_{i+a}$ for any i.*

Let $\mathcal{M}$ be an object in $\mathrm{ModFI}^{r,\varphi,N}_{/S}$. Putting $\hat{A}_{\mathrm{st},\infty} := \hat{A}_{\mathrm{st}} \otimes_{W(k)} K_0/W(k)$, we define

$$T^*_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}_{'\mathrm{Mod}^{r,\varphi,N}_{/S}}(\mathcal{M}, \hat{A}_{\mathrm{st},\infty}).$$

If $\mathcal{M} \in \mathrm{ModFI}^{r,\varphi,N}_{/S}$ is isomorphic to $S/p^{n_1}S \oplus S/p^{n_2}S \oplus \cdots \oplus S/p^{n_d}S$ as S-modules, then $T^*_{\mathrm{st}}(\mathcal{M})$ is isomorphic to $\mathbb{Z}_p/p^{n_1}\mathbb{Z}_p \oplus \mathbb{Z}_p/p^{n_2}\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p/p^{n_d}\mathbb{Z}_p$ as $\mathbb{Z}_p$-modules ([**Ca1**], Proposition 6.4.5).

The following is one of main result of [**Ca1**].

THEOREM 2.21 ([**Ca1**], Théorème 1.0.3 and 5.2.2). *Suppose that k is algebraically closed. Let $\mathcal{M}$ be a simple object of $\mathrm{ModFI}^{r,\varphi,N}_{/S}$ and $\mathcal{M}(\mathfrak{n})$ the object corresponding to $\mathcal{M}$ via the functor T (see Proposition 2.16). Let h be the period of the sequence $\mathfrak{n} = (n_i)_i$. Then the tame inertia weights of $T^*_{\mathrm{st}}(\mathcal{M})$ are $er-n_1, er-n_2, \ldots, er-n_h$. In particular, the tame inertia weights of $T^*_{\mathrm{st}}(\mathcal{M})$ are between 0 and er.*

6. Caruso's bound on tame inertia weights

The tame inertia weights of an p-adic semistable Galois representation of G_{K_λ} with Hodge-Tate weights in $[0, r]$ have remarkable properties if $er < p-1$. For example, Serre conjectured in [**Se4**] that the tame inertia weights on the Jordan-Hölder quotients of a residual representation of the r-th p-adic étale cohomology group $H^r_{\mathrm{\acute{e}t}}(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$ of a proper smooth scheme X over K are between 0 and er. Caruso proved Serre's conjecture in [**Ca2**] by using the integral p-adic Hodge theory. He also proved the analogous result on the tame inertia weights of $H^r_{\mathrm{\acute{e}t}}(X_{\bar{K}}, \mathbb{Q}_p)$ in [**Ca1**] (see also Theorem 2.24).

We fix an integer $r \ge 0$ such that $er < p-1$. We denote by $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_K)^r$ (resp. $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)_{\mathrm{tors}}$) the category of G_K-stable $\mathbb{Z}_p$-lattices of semistable p-adic representations of G_K with Hodge-Tate weights in

$[0,r]$ (resp. the category of finite torsion $\mathbb{Z}_p$-modules with a continuous G_K-action). We have already constructed the functors

$$T^*_{\mathrm{fr}} : \mathrm{Mod}^{r,\varphi,N}_{/S} \to \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_K)^r$$

and

$$T^*_{\mathrm{tors}} \colon \mathrm{ModFI}^{r,\varphi,N}_{/S} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_K)_{\mathrm{tors}}.$$

These functors have good properties.

THEOREM 2.22 ([**Ca1**], Théorème 1.0.4 and 1.0.5). (1) *The functor T^*_{fr} is an isomorphism.*
(2) *The functor T^*_{tors} is exact and fully faithful, and its essential image is stable under taking sub-objects and quotient objects.*

REMARK 2.23. In this remark, we ignore the assumption $er < p-1$. Breuil conjectured in [**Br4**] that, if $r < p-1$, the functor T^*_{fr} induces an equivalence of categories between $\mathrm{Mod}^{r,\varphi,N}_{/S}$ and $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_K)^r$. If $r \leq 1$, then the conjecture has been proved by Breuil in [**Br3**] and [**Br4**]. If $e = 1$, then Breuil showed the conjecture in [**Br2**] by generalizing arguments of Fontaine and Laffaille [**FL**]. The conjecture in the case $er < p-1$ is proved by Caruso as written in Theorem 2.22. In the general case, the conjecture is proved completely by Liu [**Li1**].

In fact, Liu defined $(\varphi, \hat{G})$*-modules of height* $\leq r$ in [**Li2**] and proved that, for any non-negative integer $r \geq 0$, the category of $(\varphi, \hat{G})$-modules of height $\leq r$ is equivariant to the category $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_K)^r$.

By the definition of strongly divisible lattices, we see that, for any strongly divisible lattice $\tilde{\mathcal{M}}$ and $n \geq 0$, the quotient $\tilde{\mathcal{M}}/p^n\tilde{\mathcal{M}}$ is an object of $\mathrm{ModFI}^{r,\varphi,N}_{/S}$ and the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Mod}^{r,\varphi,N}_{/S} \arrow[r, "T^*_{\mathrm{fr}}"] \arrow[d, "\mathrm{mod}\ p^n"] & \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_K)^r \arrow[d, "\mathrm{mod}\ p^n"] \\
\mathrm{ModFI}^{r,\varphi,N}_{/S} \arrow[r, "T^*_{\mathrm{tors}}"] & \mathrm{Rep}_{\mathbb{Z}_p}(G_K)_{\mathrm{tors}}.
\end{tikzcd}$$

Now we show the following important theorem[7]:

THEOREM 2.24 ([**Ca1**]). *Let $T_p \in \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_K)^r$ and $\bar{T}_p = T_p/pT_p$ its residual representation. Then the tame inertia weights of $\bar{T}_p|_I$ are between 0 and er.*

PROOF. We may assume that k is algebraically closed. Choose a strongly divisible lattice $\tilde{\mathcal{M}}$ corresponding to T_p via T^*_{fr}. Then $\mathcal{M} := \tilde{\mathcal{M}}/p\tilde{\mathcal{M}}$ is contained in $\mathrm{ModFI}^{r,\varphi,N}_{/S}$ and $T^*_{\mathrm{tors}}(\mathcal{M})$ is isomorphic to $\bar{T}_p$. We identify $T^*_{\mathrm{tors}}(\mathcal{M})$ with $\bar{T}_p$. Since the essential image of $T^*_{\mathrm{tors}} \colon \mathrm{ModFI}^{r,\varphi,N}_{/S} \to$

[7]Theorem 2.24 seems to be well-known for experts, but there is no proof in [**Ca1**], explicitly.

$\mathrm{Rep}_{\mathbb{Z}_p}(G_K)_{\mathrm{tors}}$ is stable under sub-quotient, any Jordan-Hölder quotient of $\bar{T}_p$ is isomorphic to a representation of the form $T^*_{\mathrm{tors}}(\mathcal{M}')$ for some $\mathcal{M}' \in \mathrm{ModFI}^{r,\varphi,N}_{/S}$. The object $\mathcal{M}'$ is simple because the functor T^*_{tors} is exact (cf. Theorem 2.22) and the image under T^*_{tors} of a non-zero object is still non-zero. Therefore, we obtain the desired result by Theorem 2.21. □

REMARK 2.25. In fact, we do not need the assumption $er < p-1$ in Theorem 2.24 (the case $er \geq p-1$ is trivial).

7. Hodge polygons, Newton polygons and tame inertia polygons

We recall three polygons associated with p-adic Galois representations (cf. [**CS**], [**Fo1**]) and consider representations whose tame inertia polygon (defined below) is a line.

The polygon associated with rational numbers $n_1 \leq n_2 \leq \cdots \leq n_d$ is the polygon with break points $(0,0)$ and $(j, n_1 + \cdots + n_j)$ for $1 \leq j \leq d$ in the usual Cartesian plane. We denote this polygon by $P(n_1, n_2, \ldots, n_d)$. For a p-adic representation V of G_K of dimension n, put $D := D^*_{\mathrm{st}}(V) := (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V^\vee)^{G_K}$ and $D_K := D \otimes_{K_0} K$, where B_{st} is Fontaine's p-adic period ring (cf. [**Fo3**]). Then D has a structure of a filtered (φ, N)-module by a natural manner. If V is semistable, then V is crystalline if and only if the monodromy operator N of $D^*_{\mathrm{st}}(V)$ is 0.

DEFINITION 2.26. The *Hodge polygon of* V is the polygon associated with integers t such that $\mathrm{Fil}^t D_K \neq \mathrm{Fil}^{t+1} D_K$ with the rule that t appears d_t-times, where $d_t := \dim_{K_0} \mathrm{Gr}^t D_K = \dim_{K_0} \mathrm{Fil}^t D_K / \mathrm{Fil}^{t+1} D_K$.

Note that, if V is a Hodge-Tate representation and denote all the Hodge-Tate weights of V by $h_1 \leq h_2 \leq \cdots \leq h_n$, then the Hodge polygon of V coincides with the polygon $P(h_1, h_2, \ldots, h_n)$. Recall that, if V is a Hodge-Tate representation, *Hodge-Tate weights of* V are the integers t such that there exists an isomorphism

$$V \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \oplus_t \mathbb{C}_p(t)^{d_t}$$

of G_K-modules with the rule that t appears d_t-times.

Let V be any p-adic representation of G_K and $D := D^*_{\mathrm{st}}(V)$ the filtered (φ, N)-module as above. For a rational number α, denote by D_α the slope α part of D associated with the action of the Frobenius of D. We have $D = \oplus_{\alpha \in \mathbb{Q}} D_\alpha$. It can be seen $\alpha d_\alpha \in \mathbb{Z}$, where $d_\alpha := \dim_{K_0} D_\alpha$.

DEFINITION 2.27. The *Newton polygon of* V is the polygon associated with the rational numbers α such that $D_\alpha \neq 0$ with the rule that α appears d_α-times.

Denote by $w_1 \leq w_2 \leq \cdots \leq w_n$ all the tame inertia weights of a p-adic representation V (cf. Definition 2.3).

DEFINITION 2.28. The *tame inertia polygon of* V is the polygon $P(t_1, t_2, \ldots, t_n)$, where $t_i := w_i/e$ for $1 \leq i \leq n$.

The above three polygons associated with V have remarkable relations.

PROPOSITION 2.29 ([**Fo1**], [**CS**]). *Suppose that* V *is semistable.*
(1) *The Newton polygon of* V *lies above the Hodge polygon of* V *and these polygons end up at the same point.*
(2) *Suppose* $er < p-1$. *Then the tame inertia polygon of* V *lies above the Hodge polygon of* V *and these polygons end up at the same point.*
(3) *Suppose* $er < p-1$. *Let* $(h_i)_{i=1,\ldots,n}, (\alpha_i)_{i=1,\ldots,n}, (t_i)_{i=1,\ldots,n}$ *be increasing sequences associated with the Hodge polygon, the Newton polygon and the tame inertia polygon of* V, *respectively. Take an integer* $1 \leq j \leq n$. *If* $h_i = \alpha_i$ *for* $1 \leq i \leq j$, *then* $t_i = h_i = \alpha_i$ *for* $1 \leq i \leq j$.

If V is semistable and the Hodge polygon of V is a line, then it is well-known that $V|_I$ is isomorphic to a Tate twist of $\mathbb{Q}_\ell^{\oplus n}$. We are interested in the case where the tame inertia polygon of V is a line, that is, V is of uniform tame inertia weight w for some integer w. Note that V is of uniform tame inertia weight w if and only if the action of I_K on a residual representation of V, with a suitable choice of basis, has the upper-triangular form of diagonal components θ_1^w because $\theta_h^{1+p+\cdots+p^{h-1}} = \theta_1$ for any integer $h \geq 0$. If $er < p-1$ and V is semistable with Hodge-Tate weights in $[0, r]$ and is of uniform tame inertia weight w, then $0 \leq w \leq er$ by Theorem 2.24. For such V, we see $w \in (e/n)\mathbb{Z} \cap \mathbb{Z}$ by Proposition 2.29 (2). If V is semistable with all the Hodge-Tate weights h, then $V(-h)|_{I_K}$ is trivial and thus $V(-h)$ is of uniform tame inertia weight 0.

EXAMPLE 2.30 (The case where $e = 1$ and V is crystalline). Suppose $e = 1$. Let V be an n-dimensional crystalline p-adic representation of G_K with Hodge-Tate weights in $[0, p-1)$. Then the Fontaine-Laffaille theory ([**FL**]) implies that the Hodge-Tate weights of V coincide with the tame inertia weights of V. Hence, for a finite-dimensional crystalline p-adic representation V of G_K with Hodge-Tate weights in $[0, p-1)$, the following are equivalent:

(1) V is of uniform tame inertia weight w;
(2) V is of all the Hodge-Tate weights w;
(3) I acts on $V(-w)$ trivially.

If the Hodge polygon or the Newton polygon of a semistable p-adic representation V of G_K is a line, then V is crystalline. In fact, under some assumptions, an analogous result holds for the case where the tame inertia polygon of V is a line.

PROPOSITION 2.31. *Suppose* $er < p-1$. *Let* V *be an* n*-dimensional semistable* p*-adic representation of* G_K *with Hodge-Tate weights in* $[0, r]$. *Assume that* V *is of uniform tame inertia weight. Then* V *is crystalline in any one of the following case:*

(1) *$r \leq 1$.*
(2) *$n = 2$ and $r \leq 2$.*
(3) *$n = 2$ and e is odd.*
(4) *$n = 3$, $r \leq 2$ and e is coprime to 3.*

PROOF. It is enough to prove that $N = 0$ on $D := D^*_{\mathrm{st}}(V)$. Denote by φ the Frobenius operator of D. For simplicity, we denote by $P_H(V)$, $P_N(V)$ and $P_T(V)$ the Hodge polygon, the Newton polygon and the tame inertia polygon associated with V, respectively. We denote by w the uniform tame inertia weight of V. Note that we have $0 \leq w \leq er$ and $w \in (e/n)\mathbb{Z} \cap \mathbb{Z}$ by Theorem 2.24 and Proposition 2.29 (2).

First we suppose the condition (1) holds. If $w = 0$, then Proposition 2.29 implies that $P_N(V)$ is the line of slope 0 and thus $N = 0$ (it follows from the relation $N(D_\alpha) \subset D_{\alpha-1}$ for any $\alpha \in \mathbb{Q}$). Thus we may suppose $w \neq 0$. Now assume $N \neq 0$. In this case there is an integer s such that s and $s+1$ are parts of slopes of $P_N(V)$. We see $s > 0$ by the condition $w \neq 0$ and Proposition 2.29. Moreover, Proposition 2.29 (1) implies that $P_H(V)$ has a slope s' such that $s' \geq s + 1$. Hence we see that $P_H(V)$ has a slope greater than 1. This contradicts the assumption $r \leq 1$.

Next we suppose the condition (2) holds and $w > 0$. Then we have $w = e/2, e, 3e/2$ or $2e$. Suppose $w = e/2$ or e. Since $P_T(V)$ ends up at the coordinate (2,1) or (2,2), we see that all the possibilities of $P_N(V)$ are $P(0,1)$, $P(0,2)$ or a line. If $P_N(V) = P(0,1)$ or $P(0,2)$, Proposition 2.29 shows that $P_H(V)$ and $P_T(V)$ are of the form $P(0,*)$. This is a contradiction. Hence $P_N(V)$ is a line and this implies $N = 0$. Suppose $w = 3e/2$. Then we see $P_H(V) = P(1,2)$. Thus $P_N(V)$ is $P(1,2)$ or the line of slope $3/2$ by Proposition 2.29 (1). Hence Proposition 2.29 (3) implies that $P_N(V)$ must be the line of slope $3/2$ and therefore, we see $N = 0$. Suppose $w = 2e$ and $N \neq 0$. Clearly $P_N(V)$ is not a line and hence $P_N(V) = P(t, 4-t)$ for some integer $0 \leq t \leq 1$. Since $N(D_\alpha) \subset D_{\alpha-1}$ for any $\alpha \in \mathbb{Q}$ and $N \neq 0$, we have $D_{(4-t)-1} = D_t$. Thus $t = 3/2$. This is a contradiction and we obtain $N = 0$.

Suppose we are in the condition (3). In this case we see $w \in (e/2)\mathbb{Z} \cap \mathbb{Z} = e\mathbb{Z}$ and $P_N(V) = P(u, 2w/e - u)$ for some integer $0 \leq u \leq w/e$. Assume $N \neq 0$. Then we have $D_{2w/e-u-1} = D_u$ and thus $u = w/e - 1/2$. This is a contradiction.

Finally, suppose we are in the condition (4). Then $w \in (e/3)\mathbb{Z} \cap \mathbb{Z} = e\mathbb{Z}$. If $w = 0$, then the assertion (1) implies V is crystalline. Suppose $w = e$. Since $P_T(V)$ ends up at the coordinate $(3,3)$, we see that all the possibilities of $P_N(V)$ are $P(0,3/2,3/2)$, $P(0,0,3)$, $P(0,1,2)$, $P(1,1,1)$ and $P(1/2,1/2,2)$. If $P_N(V) = P(0,3/2,3/2)$, $P(0,0,3)$ or $P(0,1,2)$, we know that $P_H(V)$ is of the form $P(0,*,*)$ by Proposition 2.29 (1). Hence Proposition 2.29 (3) implies that $P_T(V)$ must be also of the form $P(0,*,*)$, however, this is a contradiction. If $P_N(V) = P(1,1,1)$ or $P(1/2,1/2,2)$, then $N = 0$ because $N(D_\alpha) \subset D_{\alpha-1}$ for any $\alpha \in$

$\mathbb{Q}$. Assume $w/e \geq 2$. Since $P_T(V)$ ends up at the point $(3, 3w/e)$, Proposition 2.29 (2) and the condition $r \leq 2$ implies that $P_H(V)$ must be $P(2,2,2)$. Thus $P_N(V)$ is a line. This implies $N = 0$. $\square$

We finish this section with raising the question below:

QUESTION 2.32. *Suppose $er < p-1$. Let V be a semistable p-adic representation of G_K with Hodge-Tate weights in $[0, r]$. If V is of uniform tame inertia weight, then is V crystalline?*

CHAPTER 3

Abelian varieties

In this chapter, we recall some basic notions related with abelian varieties. At this moment, we can say that abelian varieties are classical objects by an appearance of the word *motive*. However, these are very important objects to study in algebraic number theory even now, and there exist various curious unsolved conjectures.

Definition 3.1. (1) A group scheme A over a scheme S is an *abelian scheme over S* if A is smooth and proper with geometrically irreducible fibers over S. If S is the spectrum of a field F, an abelian scheme over S is called an *abelian variety* defined over F. In particular, an *elliptic curve* is an abelian variety of dimension one.
(2) A *torus T* over a scheme S is a commutative group scheme over S such that locally on S_{fppf} (equivalently, $S_{\text{ét}}$ or S_{fpqc}) it is isomorphic to the product of finitely many copies of the multiplicative group scheme $\mathbb{G}_m$. A *split torus* is an S-scheme which is isomorphic to the product of finitely many copies of the multiplicative group over S.
(3) A group scheme G over a scheme S is a *semi-abelian scheme* if A is smooth separated commutative group scheme over S with geometrically connected fibers, such that the fiber $G_s := G \times_S \mathrm{Spec}(k(s))$ is an extension of an abelian variety A_s by a torus T_s for each $s \in S$, that is, $0 \to T_s \to G_s \to A_s \to 0$ is exact.

Definition 3.2. Let F be a field, v a discrete valuation of F, $\mathcal{O}_v$ the integer ring of v and $\mathbb{F}_v$ the residue field of v. Let A be an abelian variety over F.
(1) We say that A has *good reduction* (resp. *semistable reduction*) at v if there exists an abelian scheme (resp. semi-abelian scheme) A_v over $\mathrm{Spec}(\mathcal{O}_v)$ such that A is isomorphic to $A_v \otimes_{\mathcal{O}_v} \mathrm{Spec}(F)$ over F. If A has good reduction (resp. semistable reduction) at an extension of v to a finite algebraic extension of F, we say that A has *potentially good reduction* (resp. *potentially semistable reduction*) at v.
(2) Let the notation be as in (1) and assume A has good reduction. We call $\tilde{A} := A_v \otimes_{\mathcal{O}_v} \mathrm{Spec}(\mathbb{F}_v)$ *the reduction of the abelian variety A at v*. Note that $\tilde{A}$ is also an abelian variety over $\mathbb{F}_v$. A reduction $\tilde{A}$ of an abelian variety A is uniquely determined by A, up to canonical isomorphism.

Let A be an abelian variety over a field F with dimension d. Let $p \geq 0$ be the characteristic of F. It is well known that there are canonical isomorphisms

$$A(F^{\mathrm{sep}})[\ell^n] \simeq (\mathbb{Z}/\ell^n\mathbb{Z})^{2d}$$

for any prime $\ell \neq p$ (cf. [**GM**], Prop. 5.11). If $\ell = p > 0$,

$$A(F^{\mathrm{sep}})[p^n] \simeq (\mathbb{Z}/p^n\mathbb{Z})^{f}$$

for some integer $0 \leq f = f(A) \leq d$, which is also independent of n and $f(A)$ is called the *p-rank* of A. If two abelian varieties A and B over F are isogenous, then we have $f(A) = f(B)$. (cf. [**GM**], Prop. 5.22).

For any prime ℓ, consider the projective system

$$A(F^{\mathrm{sep}})[\ell] \xleftarrow{[\ell]} A(F^{\mathrm{sep}})[\ell^2] \xleftarrow{[\ell]} \cdots \xleftarrow{[\ell]} A(F^{\mathrm{sep}})[\ell^n] \xleftarrow{[\ell]} A(F^{\mathrm{sep}})[\ell^{n+1}] \xleftarrow{[\ell]} \cdots .$$

Definition 3.3. For any prime number ℓ, put

$$T_\ell(A) = \varprojlim_n A(F^{\mathrm{sep}})[\ell^n],$$

with respect to the above projective system. We call $T_\ell(A)$ *the Tate module of* A. We also denote the module $T_\ell(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by $V_\ell(A)$.

The Tate module $T_\ell(A)$ is a free $\mathbb{Z}_\ell$-module of finite rank, and equipped with a continuous action of G_F. If $\ell \neq p$, the $\mathbb{Z}_\ell$-rank of $T_\ell(A)$ is $2d$, whereas if $\ell = p > 0$, the $\mathbb{Z}_\ell$-rank of $T_\ell(A)$ can be taken to any number among 0 to d. Note that there exists an isomorphism

$$T_\ell(A) \simeq H^r_{\text{ét}}(A_{\bar{F}}, \mathbb{Q}_\ell)^\vee$$

as $\mathbb{Q}_\ell[G_F]$-modules.

Definition 3.4. (1) An abelian variety A of dimension d over a field F of characteristic p is *ordinary* if $f(A) = d$, that is,

$$A(F^{\mathrm{sep}})[p^n] = (\mathbb{Z}/p^n\mathbb{Z})^{\oplus d}.$$

(2) Let $\mathcal{A}$ be an abelian scheme over a valuation ring $\mathcal{O}$ and $\mathbb{F}$ the residue field of $\mathcal{O}$ with characteristic $p > 0$. We call $\mathcal{A}$ is *ordinary* if its special fiber $\mathcal{A} \times_{\mathcal{O}} \mathbb{F}$ is ordinary.
(3) Let F be a field, v a discrete valuation of F, and $\mathcal{O}_v$ the valuation ring of v with residual characteristic $p > 0$. Let A be an abelian variety over F which has good reduction at v and A_v the abelian scheme over $\mathcal{O}_v$ with its generic fiber A. We say that A has *ordinary good reduction* at v if A_v is ordinary.

For an abelian variety A over a field F, we can consider $A^\vee := \mathrm{Pic}^0_A$ as an abelian variety over F, which is called the *dual abelian variety* of A. About the dual abelian variety, following are well-known:

(1) The dimension of $A^\vee$ is the same as the dimension of A.
(2) The abelian varieties A and $A^\vee$ are isogenous over $\bar{F}$.

(3) For any prime number ℓ, there is a canonical isomorphism
$$A^\vee[\ell^n] \simeq A[\ell^n]^\vee$$
as F-group schemes, where $A[\ell^n]^\vee$ is the Cartier dual of $A[\ell^n]$. Hence if ℓ is different from the characteristic of F, we see that
$$T_\ell(A^\vee) \simeq T_\ell(A)^\vee(1)$$
as G_F-modules.

The next proposition is well-known, which is called *the criterion of Néron-Ogg-Shafarevich*.

Proposition 3.5 ([**ST**], Thm. 1). *Let F be a field, v a discrete valuation of F and $\mathbb{F}_v$ a residue field of F at v. For any abelian variety A over F, the following properties are equivalent.*

(1) *A has good reduction at v.*
(2) *$A(F^{\mathrm{sep}})[m]$ is unramified at v for all m prime to characteristic* $\mathrm{char}(\mathbb{F}_v)$ *of $\mathbb{F}_v$.*
(3) *There exist infinitely many integers m, which is prime to* $\mathrm{char}(\mathbb{F}_v)$*, such that $A(F^{\mathrm{sep}})[m]$ is unramified at v.*
(4) *The Tate module $T_\ell(A)$ is unramified at v for some (all) primes $\ell \neq \mathrm{char}(\mathbb{F}_v)$.*

It is well-known that there exist only finitely many number fields with bounded degree and given ramification set (Hermite-Minkowski Theorem). An analogous result on the above fact for abelian varieties is known.

Theorem 3.6 ([**Fa**]). *Let K be a finite extension of $\mathbb{Q}$ and S a finite set of finite places of K. Then there exists only finitely many K-isomorphism classes of abelian varieties which have good reduction outside S.*

Let A be a g-dimensional abelian variety over a field F. We denote by $\mathrm{End}_F(A)$ the ring of F-endomorphisms of A. Let E be a finite extension of $\mathbb{Q}$ of degree $2g$, and let $i\colon E \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_F(A)$ be a ring homomorphism. We call the pair (A, i) an *abelian variety with complex multiplication by E over the field F*.

Proposition 3.7 ([**ST**], Section 4, Corollary 2). *Let A be an abelian variety with complex multiplication over a field F. Let $\rho_{A,\ell}$ be the representation $G_F \to GL(T_\ell(A))$ defined by the Tate-module of A. Then $\rho_{A,\ell}$ is abelian.*

Let K be a finite extension of $\mathbb{Q}$. Let A be an abelian variety over K. Suppose that the representation $\rho_{A,\ell}\colon G_F \to GL(T_\ell(A))$ defined by the Tate-module of A is abelian. Then the representation $\rho_{A,\ell}$ is $\mathbb{Q}$-rational, semisimple and locally algebraic, and thus, by Theorem 1.17, we see that $\rho_{A,\ell}$ arises from an algebraic homomorphism $\phi\colon S_{\mathfrak{m}} \to$

GL_{2g} defined over $\mathbb{Q}$ (see Section 2.4). Since $\rho_{A,\ell'}$ is isomorphic to the representation $\phi_{\ell'}$ arising from ϕ for all prime numbers ℓ', we have that $\rho_{A,\ell'}$ is abelian for any prime ℓ'. Therefore, we obtain the following.

PROPOSITION 3.8. *Let K be a finite extension of $\mathbb{Q}$. Let A be an abelian variety over K. For any prime number ℓ, we denote by $\rho_{A,\ell}$ the ℓ-adic representation of G_K defined by the Tate-module of A. Then $\rho_{A,\ell}$ is abelian for some prime number ℓ if and only if $\rho_{A,\ell}$ is abelian for all prime numbers ℓ.*

CHAPTER 4

Modular curves

In this chapter, we are interested in a study of a structure of the set of rational points on elliptic curves. For this study, modular curves have been used by many mathematicians. The points of a modular curve parametrize isomorphism classes of elliptic curves, together with some additional structure. Let $N \geq 2$ be a positive integer. It is known that there exist smooth irreducible affine curves $Y_0(N)$ and $Y_1(N)$ defined over $\mathbb{Q}$ which are coarse moduli spaces $Y_0(N)$ and $Y_1(N)$ for elliptic curves with $\Gamma_0(N)$- and $\Gamma_1(N)$-structures, respectively. In fact, $Y_1(N)$ is a fine moduli scheme for $N \geq 5$. For any finite extension K of $\mathbb{Q}$, we regard K-rational points of $Y_0(N)$ and $Y_1(N)$ as follows:

- $Y_0(N)(K) := \{(E, C) \mid E$ is an elliptic curve over K, and $C \subset E(\bar{K})$ is a G_K-stable cyclic subgroup of order N $\}$.
- $Y_1(N)(K) := \{(E, P) \mid E$ is an elliptic curve over K, and $P \in E(K)$ is a point of order $N\}$.

We denote by $X_i(N)$ the compactification of $Y_i(N)$ for each i and we call $X_i(N)$ a *modular curve of level* N for each i. Points on $X_i(N) \smallsetminus Y_i(N)$ are called *cusps*. The K-rational points on modular curves, in particular non-cuspidal points, have been studied by many mathematicians. Related with the non-existence of rational points on $Y_0(N)$, the following are well-known:

Theorem 4.1. (1) ([**Ma**], Theorem 1) *Let ℓ be a prime number. Then $Y_0(\ell)(\mathbb{Q})$ is empty except for $\ell = 2, 3, 5, 7, 11, 17, 19, 37, 43, 67, 163$.* (2) ([**Mo**], Theorem B) *Let K be a quadratic field which is not an imaginary quadratic field of class number one. Then $Y_0(\ell)(K)$ is not empty only for finitely many prime numbers ℓ.*

Study of the non-existence of rational points on $Y_1(N)$ is related with the study of the torsion part $E(K)_{\mathrm{tor}}$ of $E(K)$.

Conjecture 4.2 (Torsion conjecture for abelian varieties). *If A is a g-dimensional abelian variety over a number field K, then $\sharp A(K)_{\mathrm{tor}}$ is bounded by a constant $N(g, K)$ depending only on g and K.*

Conjecture 4.3 (Strong torsion conjecture for abelian varieties). *If A is a g-dimensional abelian variety over a number field K of degree d, then $\sharp A(K)_{\mathrm{tor}}$ is bounded by a constant $N(g, d)$ depending only on g and d.*

THEOREM 4.4 ([**Me**]). *Conjecture 4.3 holds for* $g = 1$. *Moreover, we can choose* $N(1, d)$ *as* $N(1, d) = d^{3d^2}$.

We state a strengthening of Merel's result, due to Oesterlé.

THEOREM 4.5 ([**Oe**]). *If* E *is an elliptic curve defined over a number field* K *of degree* d, *and* $E(K)$ *has a point of prime order* ℓ, *then* $\ell \leq (1 + 3^{d/2})^2$.

CHAPTER 5

Rasmussen-Tamagawa Conjecture

The Shafarevich Conjecture, which is now proved by Faltings, is well-known as a conjecture on the finiteness of certain abelian varieties over a number field. In 2008, Rasmussen and Tamagawa [**RT**] conjectured the finiteness of abelian varieties with constrained prime power torsion, in the spirit of the Shafarevich Conjecture.

Throughout this section, we denote by K a finite extension of $\mathbb{Q}$ and ℓ a prime number.

1. Statement

Let $K(\mu_\ell)$ be the smallest field containing K and all ℓ-th roots of unity. We denote by $\tilde{K}_\ell$ the maximal pro-ℓ extension of $K(\mu_\ell)$ which is unramified away from ℓ.

DEFINITION 5.1. Let $g \geq 0$ be an integer. We denote by $\mathcal{A}(K, g, \ell)$ the set of K-isomorphism classes of abelian varieties A over K of dimension g which satisfy the following equivalent conditions:

(1) $K(A[\ell^\infty]) \subset \tilde{K}_\ell$;
(2) The abelian variety A has good reduction outside ℓ and the extension $K(A[\ell])/K(\mu_\ell)$ is an ℓ-extension.

The equivalence of the above conditions follows from the criterion of Néron-Ogg-Shafarevich (see Proposition 3.5). The set $\mathcal{A}(K, g, \ell)$ is finite because of the Shafarevich Conjecture. Rasmussen and Tamagawa conjectured that this set is empty for any ℓ large enough:

CONJECTURE 5.2 ([**RT**], Conjecture 1). *The set* $\mathcal{A}(K, g) := \{(A, \ell) \mid [A] \in \mathcal{A}(K, g, \ell),\ \ell : \text{prime number}\}$ *is finite, that is, the set* $\mathcal{A}(K, g, \ell)$ *is empty for any prime* ℓ *large enough.*

We call this conjecture the *Rasmussen-Tamagawa Conjecture*.

2. Structure lemma

We shall recall a lemma proved by Rasmussen and Tamagawa (cf. [**RT**], Lemma 3). Let G be a topological group with a normal pro-ℓ open subgroup N, such that the quotient $\Delta = G/N$ is isomorphic to a subgroup of $\mathbb{F}_\ell^\times$. Because N is pro-ℓ, we see that N has trivial image under any character $\psi \colon G \to \mathbb{F}_\ell^\times$. Hence, there always exists an induced character $\bar{\psi} \colon \Delta \to \mathbb{F}_\ell^\times$. Let $\chi \colon G \to \mathbb{F}_\ell^\times$ be a character such that the induced character $\bar{\chi}$ is an injection $\Delta \hookrightarrow \mathbb{F}_\ell^\times$. Finally, let $\bar{V}$ be a finite dimensional $\mathbb{F}_\ell$-vector space of dimension n on which G acts continuously.

LEMMA 5.3. *Let the notation be as above. Then the vector space $\bar{V}$ has a filtration of G-modules*

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_n = \bar{V}$$

such that $\bar{V}_k$ has dimension k for each $1 \leq k \leq n$. Furthermore, for each $1 \leq k \leq n$, the G-action on the space $\bar{V}_k/\bar{V}_{k-1}$ is given by $g.\bar{v} = \chi(g)^{a_k} \cdot \bar{v}$ for some $a_k \in \mathbb{Z}$, $0 \leq a_k < \#\Delta$.

PROOF. First we note that, for any positive integer m which divides $\ell - 1$, there exists the only one subgroup C_m of $\mathbb{F}_\ell^\times$ of order m. If we denote by δ a generator of Δ, then $\chi(\delta)$ generates $C_\Delta := C_{\sharp\Delta}$. We prove the Lemma by induction on n. Since any character $\psi\colon G \to \mathbb{F}_\ell^\times$ factors through Δ and thus has values in $C_\Delta =< \chi(\delta) >$, we obtain $\psi(\delta) = \chi^a(\delta)$ for some integer a and hence we have $\psi = \chi^a$. This finish the proof in the case $n = 1$. Assume the results holds for $\mathbb{F}_\ell$-vector space of dimension $n-1$, and let $\bar{V}$ be an $\mathbb{F}_\ell$-vector space of dimension n.

Consider the action of N on $\bar{V}$, which necessarily factors through some finite ℓ-group N_0. Hence, the N-orbits of $\bar{V}$ must all have order a power of ℓ and so the subspace $\bar{V}^N$ of fixed points is non-trivial ($\bar{V}^N = \{0\}$ implies ℓ divides $\sharp\bar{V} - 1$, which is impossible). Further, because N is normal in G, we have that $\bar{V}^N$ is G-stable, and so a well-defined action of Δ on V^N is induced. Fix a basis for $\bar{V}^N$, and let $\rho\colon \Delta \to GL(\bar{V}^N)$ be the associated representation. Put A be the matrix corresponding to $\rho(\delta)$. Since $\delta^{\sharp\Delta} = 1$, we see that the eigenvalues of A have values in C_Δ. Hence A has an eigenvector $\bar{w}$ with some eigenvalue $\lambda \in C_\Delta$. Since C_Δ is generated by $\chi(\delta)$, there exists an integer a such that $\lambda = \chi^a(\delta)$. Because ρ is a representation induced by the G-action on $\bar{V}$, we see that, under the G-action on $\bar{V}$, G acts on $\bar{w} \in \bar{V}$ by the formula $g.\bar{w} := \chi^a(g) \cdot \bar{w}$ for any $g \in G$. Therefore, $\bar{W} :=< \bar{w} >$ is a 1-dimensional G-stable subspace of $\bar{V}$ and G acts on $\bar{W}$ via χ^a. There is an induced G-action on $\bar{V}/\bar{W}$, which has dimension $n-1$. By the induction hypothesis, there exists a filtration of G-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_{n-1} = \bar{V}/\bar{W}$$

such that $\bar{V}_k$ has dimension k for each $1 \leq k \leq n-1$. Furthermore, for each $1 \leq k \leq n$, the G-action on the space $\bar{V}_k/\bar{V}_{k-1}$ is given by $g.\bar{v} = \chi(g)^{a_k} \cdot \bar{v}$ for some $a_k \in \mathbb{Z}$. By the Brauer-Nesbitt theorem, we have that $\bar{V}$ itself have a desired filtration as above.

□

COROLLARY 5.4. *Let A be a g-dimensional abelian variety over K. Then $[A] \in \mathcal{A}(K, g, \ell)$ if and only if A satisfies the following: The* *abelian variety A has good reduction outside ℓ and $A[\ell]$ has a filtration of G_K-modules*

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]$$

such that $\bar{V}_k$ has dimension k for each $1 \leq k \leq 2g$. Furthermore, for each $1 \leq k \leq 2g$, the G_K-action on the space $\bar{V}_k/\bar{V}_{k-1}$ is given by $g.\bar{v} = \chi_\ell(g)^{a_k} \cdot \bar{v}$ for some $a_k \in \mathbb{Z}$.

3. Known results

THEOREM 5.5 ([**RT**], Theorem 2 and 4). *The Rasmussen-Tamagawa Conjecture holds under any one of the following conditions:*

(i) *$K = \mathbb{Q}$ and $g = 1$.*

(ii) *K is a quadratic number field other than the imaginary quadratic fields of class number one and $g = 1$.*

PROOF. Take $(E, \ell) \in \mathcal{A}(K, 1)$. By Corollary 5.4, the group of ℓ-torsion points $E[\ell]$ of E has a G_K-stable subspace of dimension 1, which is of course cyclic of order ℓ. Recall that $Y_0(N)$ denotes the moduli space (over $\mathbb{Z}[1/N]$) for isomorphism classes of pairs (E_0, C_0), where E_0 is an elliptic curve, and C_0 is a cyclic subgroup of E_0 of order N. Then (E, ℓ) corresponds to a class $[(E, C)] \in Y_0(\ell)(K)$. If K is a number field as in the statement of this theorem, then it is known that $Y_0(\ell)(K)$ is empty for ℓ large enough (cf. Theorem 4.1). $\square$

CHAPTER 6

Non-existence of Galois representations

Let ℓ be a prime number and K a number field. In this chapter, we show the non-existence of certain semistable ℓ-adic Galois representations of the absolute Galois group G_K of K by using remarkable results on the tame inertia weights due to Caruso. Fix non-negative integers n, r and w, and a prime number $\ell_0 \neq \ell$. Put $\bullet := (n, \ell_0, r, w)$. We consider the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ of isomorphism classes of certain ℓ-adic representations of G_K (Definition 6.4 (2)). The set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ has some relations with the dual of $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$, where X is a proper smooth scheme over K which has semistable reduction everywhere and has good reduction at a place of K above ℓ_0. Our main result in this chapter is

THEOREM 6.1 (= Theorem 6.8). *Suppose that w is odd or $w > 2r$. Then there exists an explicit constant C depending only on K, n, ℓ_0, r and w such that* $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ *is empty for any prime number $\ell > C$ which does not split in K.*

We prove this theorem by a relation between tame inertia weights and Frobenius weights (Proposition 6.11). Our result gives an application to a special case of the Rasmussen-Tamagawa Conjecture (see the previous chapter).

1. Geometric and filtration conditions

We define the set of representations we mainly consider throughout this chapter. We fix non-negative integers n, r, w and $\bar{w}$, and a prime number ℓ_0 different from ℓ. Let $\bar{\chi}_\ell$ be the mod ℓ cyclotomic character. Take an n-dimensional ℓ-adic representation V of G_K and denote by $\bar{V}$ a residual representation of V. Note that $\bar{V}$ is not uniquely determined (it depends on a choice of a G_K-stable lattice), but the required property or the numbers computed from $\bar{V}$ does not depend on the choice of $\bar{V}$. Now we consider the following geometric conditions (G-1), (G-2), (G-2)$'$ and (G-3), and filtration conditions (F-1) and (F-2):

- (G-1) For any place λ of K above ℓ, the representation $V|_{G_\lambda}$ is semistable and has Hodge-Tate weights in $[0, r]$.
- (G-2) For some place v_0 of K above ℓ_0, the representation V is unramified at v_0 and the characteristic polynomial $\det(T - \mathrm{Fr}_{v_0}|V)$ has rational integer coefficients. Furthermore, there exist non-negative integers $w_1(V), w_2(V), \dots, w_n(V)$ such that

$w_1(V) + w_2(V) + \cdots + w_n(V) \leq \bar{w}$ and, for every embedding $\bar{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, the roots of the above characteristic polynomial have complex absolute values $q_{v_0}^{w_1(V)/2}, q_{v_0}^{w_2(V)/2}, \ldots, q_{v_0}^{w_n(V)/2}$.

(G-2)$'$ The condition (G-2) holds and $w_1(V) = w_2(V) = \cdots = w_n(V) = w$.

(G-3) For any finite place v of K not above ℓ, the action of I_v on $\bar{V}$ is unipotent.

(F-1) The representation $\bar{V}$ has a filtration of G_K-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_n = \bar{V}$$

such that $\bar{V}_k$ has dimension k for each $1 \leq k \leq n$.

(F-2) The condition (F-1) holds. Moreover, for each $1 \leq k \leq n$, the G_K-action on the quotient $\bar{V}_k/\bar{V}_{k-1}$ is given by $g.\bar{v} = \bar{\chi}_\ell^{a_k}(g)\bar{v}$ for some $0 \leq a_k \leq \ell - 2$.

If an ℓ-adic representation V satisfies the condition (F-1), then we say that V is *of residually Borel*. If $n = 2$, then (F-1) is equivalent to the condition that $\bar{V}$ is reducible.

EXAMPLE 6.2. Suppose $w \leq r$. Let X be a proper smooth scheme over K which has everywhere semistable reduction and has good reduction at some places of K above ℓ_0. Then the dual $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)^\vee$ of the w-th ℓ-adic étale cohomology group of X satisfies the geometric conditions (G-1), (G-2)$'$ and (G-3).

PROPOSITION 6.3. *Let X be a proper smooth scheme over K and w an odd integer. Denote by S_X the finite set of prime numbers p such that X has bad reduction at some place of K above p. Then, there exists a finite extension L of K such that, for any $\ell \notin S_X$, the ℓ-adic representation $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)$ of G_L is semistable at all finite places.*

In particular, we have the following: Let X and L be as above. Fix a prime number $\ell_0 \notin S_X$ and take a prime number ℓ such that $\ell \neq \ell_0$ and $\ell \notin S_X$. Then $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)^\vee$ satisfies (G-1), (G-2)$'$ and (G-3) as a representation of G_L.

PROOF OF PROPOSITION 6.3. For any algebraic extension K' of K, denote by $S_{X,K'}$ the set of places of K' which is above one of the prime numbers in S_X. Take any place $v \in S_{X,K}$. By de Jong's alteration theorem ([**dJ**], Theorem 6.5), there exist a finite extension K'_v of K_v, a proper strictly semistable scheme $\mathcal{Y}^v$ over $\mathcal{O}_{K'_v}$ and a morphism $\mathcal{Y}^v \to \mathcal{X}$ compatible with $\mathrm{Spec}(\mathcal{O}_{K'_v}) \to \mathrm{Spec}(\mathcal{O}_{K_v})$ such that the morphism $f\colon \mathcal{Y}^v \to \mathcal{X}_{\mathcal{O}_{K'_v}}$ induced by the above morphism is an étale alteration (see also [**Ts1**], Theorem A3). Here $\mathcal{O}_{K_v}$ and $\mathcal{O}_{K'_v}$ are integer rings of K_v and K'_v, respectively, and $\mathcal{X}$ is a proper flat model of X_{K_v} over $\mathcal{O}_{K_v}$. Such a model always exists by the compactification theorem of Nagata. Take any prime number ℓ'. If we denote by f_* and f^* the induced homomorphisms $H^w_{\text{ét}}(\mathcal{Y}^v_{\bar{K}'_v}, \mathbb{Q}_{\ell'}) \to H^w_{\text{ét}}(X_{\bar{K}_v}, \mathbb{Q}_{\ell'})$

and $H^w_{\text{ét}}(X_{\bar{K}_v},\mathbb{Q}_{\ell'}) \to H^w_{\text{ét}}(\mathcal{Y}^v_{\bar{K}'_v},\mathbb{Q}_{\ell'})$ respectively, then the map $f_* \circ f^*$ is the multiplication by $\deg(f)$. In particular, the map f^* is injective. Thus we may consider that $H^w_{\text{ét}}(X_{\bar{K}'_v},\mathbb{Q}_{\ell'})$ is a sub-representation of $H^w_{\text{ét}}(\mathcal{Y}^v_{\bar{K}'_v},\mathbb{Q}_{\ell'})$. Now take a finite extension $K(v)$ of K and a place $w(v)$ of $K(v)$ above v such that $K(v)_{w(v)} = K'_v$, where $K(v)_{w(v)}$ is the $w(v)$-adic completion of $K(v)$. The existence of $K(v)$ and $w(v)$ is an easy consequence of [**La**], Section II, Section 2, Proposition 4. We denote by L the Galois closure, over K, of the field generated by all $K(v)$. Here v runs through all the places of K in $S_{X,K}$. Now we take a prime number $\ell \notin S_X$. It suffices to show that the ℓ-adic representation $H^w_{\text{ét}}(X_{\bar{L}},\mathbb{Q}_\ell)$ of G_L is everywhere semistable. Take any finite place w_L of L. If $w_L \notin S_{X,L}$, then X has good reduction at w_L and in particular $H^w_{\text{ét}}(X_{\bar{L}},\mathbb{Q}_\ell)$ is semistable at w_L. Suppose $w_L \in S_{X,L}$. We denote the restriction of w_L to K by v. Take $\mathcal{Y}^v$ and the place $w(v)$ of $K(v)$ as above. Furthermore, we take a place w'_L of L above $w(v)$. Since the action of $I_{w'_L}$ is unipotent on $H^w_{\text{ét}}(\mathcal{Y}^v_{\bar{L}},\mathbb{Q}_\ell)$, we have that the action of $I_{w'_L}$ on $H^w_{\text{ét}}(X_{\bar{K}},\mathbb{Q}_\ell)$ is unipotent, too. Since the inertia subgroup $I_{w'_L}$ conjugates with I_{w_L} by the element of G_K, we see that the action of I_{w_L} on $H^w_{\text{ét}}(X_{\bar{K}},\mathbb{Q}_\ell)$ is also unipotent, that is, $H^w_{\text{ét}}(X_{\bar{L}},\mathbb{Q}_\ell)$ is semistable at w_L. This finishes the proof. □

Definition 6.4. Put $\circ := (n,\ell_0,r,\bar{w})$ and $\bullet := (n,\ell_0,r,w)$.
(1) We denote by $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ_{\mathrm{cycl}}$ (resp. $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet_{\mathrm{cycl}}$) the set of isomorphism classes of n-dimensional ℓ-adic representations V of G_K which satisfy (G-1), (G-2) and (F-2) (resp. (G-1), (G-2)$'$ and (F-2)).
(2) We denote by $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ$ (resp. $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$) the set of isomorphism classes of n-dimensional ℓ-adic representations V of G_K which satisfy (G-1), (G-2), (G-3) and (F-1) (resp. (G-1), (G-2)$'$, (G-3) and (F-1)).

Clearly, we have

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ_{\mathrm{cycl}} & \subset & \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ \\ \cup & & \cup \\ \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet_{\mathrm{cycl}} & \subset & \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet, \end{array}$$

where $\bullet = (n,\ell_0,r,w)$ and $\circ = (n,\ell_0,r,\bar{w})$ for any $nw \le \bar{w}$.

Our main concern in this chapter is the following question:

Question 6.5. *Does there exist a constant C which depends on K and $\bullet$ (or $\circ$) such that the sets defined in Definition 6.4 are empty for $\ell > C$? If the answer is positive, how can we evaluate such a constant C?*

Remark 6.6 (Trivial case). Take a representation $V \in \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$. By (G-2)$'$, the complex absolute value of the determinant of Fr_{v_0} acting on V is $q_{v_0}^{nw/2}$ and this must be an integer. From this fact, if n

and w are odd and the extension $K/\mathbb{Q}$ is Galois of an odd degree, then $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ is empty for any prime $\ell \neq \ell_0$. As this example, there exist lots of pairs of $(K, \bullet)$ (resp. $(K, \circ)$) such that $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ (resp. $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ$) is empty for a prime ℓ (large enough). We hope to know "non-trivial cases" of the emptiness of the sets given in Definition 6.4.

2. Main theorems

We denote by d, d_K and h_K^+ the extension degree of K over $\mathbb{Q}$, the discriminant of K and the narrow class number of K, respectively. Put $M := \max\{nr, \bar{w}/2\}$ and

$$c_n := \begin{cases} \binom{n}{n/2} & \text{if } n \text{ is even,} \\ \binom{n}{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

Clearly this is equal to $\max\{\binom{n}{m} \mid 0 \le m \le n\}$. Now we put

$$\begin{aligned} &\varepsilon_1 := dM, \quad \varepsilon_2 := d\varepsilon_1, \quad \varepsilon_1' := dh_K^+ M, \quad \varepsilon_2' := d\varepsilon_1', \\ &C_1 := C_1(d, \bullet) := 2c_n\ell_0^{\varepsilon_1}, \quad C_2 := C_2(d, \bullet) := 2c_n\ell_0^{\varepsilon_2}, \\ &C_1' := C_1'(K, \bullet) := 2c_n\ell_0^{\varepsilon_1'}, \quad C_2' := C_2'(K, \bullet) := 2c_n\ell_0^{\varepsilon_2'}. \end{aligned}$$

The following are our main results.

THEOREM 6.7. *The set* $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet_{\mathrm{cycl}}$ *is empty under any one of the following situations:*

(a) *w is odd, $\ell \nmid d_K$ and $\ell > C_1$;*
(b) *w is odd, the extension $K/\mathbb{Q}$ has odd degree and $\ell > C_2$;*
(c) *$w > 2r$, $\ell \nmid d_K$ and $\ell > C_1$;*
(d) *$w > 2r$ and $\ell > C_2$;*
(e) *w and n are odd, and $\ell > C_2$.*

THEOREM 6.8. *If ℓ does not split in K, then the set* $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ *is empty under any one of the following situations:*

(a) *w is odd, $\ell \nmid d_K$ and $\ell > C_1'$;*
(b) *w is odd, the extension $K/\mathbb{Q}$ has odd degree and $\ell > C_2'$;*
(c) *$w > 2r$, $\ell \nmid d_K$ and $\ell > C_1'$;*
(d) *$w > 2r$ and $\ell > C_2'$;*
(e) *w and n are odd, and $\ell > C_2'$.*

It is useful to give the following definition for the proofs of the above theorems.

DEFINITION 6.9. Let λ be a place of K above ℓ and V an ℓ-adic representation of G_K. The *tame inertia weights of V at λ* is the tame inertia weights of $V|_{G_\lambda}$ (cf. Definition 2.3). For an integer $0 \le w < \ell - 1$, we say that V is *of uniform tame inertia weight w at λ* if $V|_{G_\lambda}$ is of uniform tame inertia weight w (cf. Definition 2.4).

The following two propositions play an essential role for our main results.

PROPOSITION 6.10. *Any ℓ-adic representation V in the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\circ}_{\mathrm{cycl}}$ has tame inertia weights $e_\lambda w_1(V)/2, e_\lambda w_2(V)/2, \ldots, e_\lambda w_n(V)/2$ at any place λ of K above ℓ under any one of the following situations:*

(a) $\ell \nmid d_K$ *and* $\ell > C_1$*;*
(b) $\ell > C_2$.

PROPOSITION 6.11. *Suppose that ℓ is a prime number which does not split in K. Any ℓ-adic representation V in the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\circ}$ has tame inertia weights $e_\lambda w_1(V)/2, e_\lambda w_2(V)/2, \ldots, e_\lambda w_n(V)/2$ at the unique place λ of K above ℓ under any one of the following situations:*

(a) $\ell \nmid d_K$ *and* $\ell > C'_1$*;*
(b) $\ell > C'_2$.

To prove these propositions, we need the following lemma:

LEMMA 6.12. *Let $s, t_1, t_2, \ldots, t_n$ and u be non-negative integers such that $0 \leq s \leq u$ and $0 \leq t_k \leq ru$ for all k. Let V be an n-dimensional ℓ-adic representation of G_K which satisfies* (G-2)*. Decompose* $\det(T - \mathrm{Fr}_{v_0}|V) = \prod_{1\leq k\leq n}(T - \alpha_k)$*. If the set $\{\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s\}$ coincides with the set $\{q_{v_0}^{t_1}, q_{v_0}^{t_2}, \ldots, q_{v_0}^{t_n}\}$ in $\bar{\mathbb{F}}_\ell$ and $\ell > 2c_n\ell_0^{dMu}$, then $\{\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s\} = \{q_{v_0}^{t_1}, q_{v_0}^{t_2}, \ldots, q_{v_0}^{t_n}\}$. In particular, we obtain*

$$\{sw_1(V)/2, sw_2(V)/2, \ldots, sw_n(V)/2\} = \{t_1, t_2, \ldots, t_n\}.$$

PROOF. We basically follow the proof by the method which has been pointed out by Rasmussen and Tamagawa. Let us denote by $S_m(x_1, x_2, \ldots, x_n)$ the elementary symmetric polynomial of degree m with n-indeterminates $x_1, x_2, \ldots, x_n$ for $0 \leq m \leq n$, that is,

$$\prod_{1\leq k\leq n}(T - x_k) = \sum_{0\leq m\leq n} S_m(x_1, x_2, \ldots, x_n)T^{n-m}.$$

For any $0 \leq m \leq n$, the condition (G-2) implies that $S_m(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a rational integer for all m and hence $S_m(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s)$, which is a symmetric polynomial of $\alpha_1, \alpha_2, \ldots, \alpha_n$, is also a rational integer. On the other hand, we have

$$\begin{aligned}|S_m(\alpha_1^s, \alpha_2^s, \ldots, \alpha_n^s)| &\leq \sum_{1\leq s_1<\cdots<s_m\leq n} (q_{v_0}^{(w_{s_1}(V)+\cdots+w_{s_m}(V))/2})^s \\ &\leq \sum_{1\leq s_1<\cdots<s_m\leq n} (q_{v_0}^{\bar{w}/2})^s = \binom{n}{m}(q_{v_0}^{\bar{w}/2})^s \leq c_n\ell_0^{dMu}\end{aligned}$$

and

$$\begin{aligned}|S_m(q_{v_0}^{t_1}, q_{v_0}^{t_2}, \ldots, q_{v_0}^{t_n})| &\leq \sum_{1\leq s_1<\cdots<s_m\leq n} q_{v_0}^{t_{s_1}+\cdots+t_{s_m}} \\ &\leq \sum_{1\leq s_1<\cdots<s_m\leq n} q_{v_0}^{nru} = \binom{n}{m} q_{v_0}^{nru} \leq c_n\ell_0^{dMu}\end{aligned}$$

by (G-2), where $|\cdot|$ is the complex absolute value. Since we have $S_m(\alpha_1^s, \alpha_2^s, \dots, \alpha_n^s) \equiv S_m(q_{v_0}^{t_1}, q_{v_0}^{t_2}, \dots, q_{v_0}^{t_n}) \bmod \ell$ and $\ell > 2c_n \ell_0^{dMu}$, we obtain

$$S_m(\alpha_1^s, \alpha_2^s, \dots, \alpha_n^s) = S_m(q_{v_0}^{t_1}, q_{v_0}^{t_2}, \dots, q_{v_0}^{t_n})$$

for all m. Therefore, we have the desired result. $\square$

Now we start the proofs of Proposition 6.10 and 6.11. Take any representation V which is an element of the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ$ and denote a residual representation of V by $\bar{V}$. Then the representation $\bar{V}$ has a filtration of G_K-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_n = \bar{V}$$

such that $\bar{V}_k$ has dimension k for each $1 \le k \le n$. We denote by $\psi_k \colon G_K \to \mathbb{F}_\ell^\times$ the character corresponding to the action of G_K on the quotient $\bar{V}_k/\bar{V}_{k-1}$ for each $1 \le k \le n$. Take any place λ of K above ℓ. By Theorem 2.24, we obtain $\psi_k = \theta_{1,v}^{b_{k,\lambda}}$ on I_λ for some integer $0 \le b_{k,\lambda} \le e_\lambda r$, where $\theta_{1,\lambda} \colon I_\lambda \to \mathbb{F}_\ell^\times$ is the fundamental character of level one at λ. Take a place λ_0 of K above ℓ_0 as in (G-2) and decompose $\det(T - \mathrm{Fr}_{\lambda_0}|V) = \prod_k (T - \alpha_k)$. Then, we see

$$\{\alpha_1, \alpha_2, \dots, \alpha_n\} = \{\psi_1(\mathrm{Fr}_{\lambda_0}), \psi_2(\mathrm{Fr}_{\lambda_0}), \dots, \psi_n(\mathrm{Fr}_{\lambda_0})\} \qquad (*)$$

in $\bar{\mathbb{F}}_\ell$.

Proof of Proposition 6.10. Assume that V is an element of the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\circ_{\mathrm{cycl}}$. Then we may suppose $\psi_k = \bar{\chi}_\ell^{a_k}$ for any k by (F-2). The relation $\bar{\chi}_\ell^{a_k} = \theta_{1,\lambda}^{b_{k,\lambda}}$ on I_λ implies $\theta_{1,\lambda}^{e_\lambda a_k} = \theta_{1,\lambda}^{b_{k,\lambda}}$ and thus $e_\lambda a_k \equiv b_{k,\lambda} \bmod \ell - 1$. Hence we have $\bar{\chi}_\ell^{e_\lambda a_k} = \bar{\chi}_\ell^{b_{k,\lambda}}$ on G_K and thus the set $\{\alpha_1^{e_\lambda}, \alpha_2^{e_\lambda}, \dots, \alpha_n^{e_\lambda}\}$ coincides with the set $\{q_{\lambda_0}^{b_{1,\lambda}}, q_{\lambda_0}^{b_{2,\lambda}}, \dots, q_{\lambda_0}^{b_{n,\lambda}}\}$ in $\bar{\mathbb{F}}_\ell$ by $(*)$. By Lemma 6.12, we have

$$\{e_\lambda w_1(V)/2, \dots, e_\lambda w_n(V)/2\} = \{b_{1,\lambda}, \dots, b_{n,\lambda}\}$$

if $\ell > 2c_n \ell_0^{dMe_\lambda}$. Since $e_\lambda \le d$ and $e_\lambda = 1$ if $\ell \nmid d_K$, we have the desired result. $\square$

Proof of Proposition 6.11. We note that each ψ_k is unramified away from ℓ by (G-3). Now we assume that any one of the following conditions (A) or (B) holds:

(A) $\ell \nmid d_K$;
(B) No additional assumptions.

Setting $b'_k := b_{k,\lambda}/e_\lambda \in \mathbb{Q}$, we have $0 \le b'_k \le r$. We note that, if we put

$$D := \begin{cases} 1 & \text{under (A)}, \\ d & \text{under (B)}, \end{cases}$$

then we see $D/e_\lambda \in \mathbb{Z}$. Since $\psi_k = \theta_{1,\lambda}^{b_{k,\lambda}}$ on I_λ, we see that $\psi_k^{e_\lambda} \bar{\chi}_\ell^{-b_{k,\lambda}}$ is trivial on I_λ and thus $(\psi_k^{e_\lambda} \bar{\chi}_\ell^{-b_{k,\lambda}})^{D/e_\lambda} = \psi_k^D \bar{\chi}_\ell^{-b'_k D}$ is also trivial on I_λ.

Since the characters ψ_k and $\bar{\chi}_\ell$ are unramified away from ℓ, this implies that $\psi_k^D \bar{\chi}_\ell^{-b'_k D}$ is unramified at all finite places of K (recall that ℓ does not split in K). By class field theory, it follows

$$\psi_k^{Dh_K^+} = \bar{\chi}_\ell^{b'_k Dh_K^+}$$

on G_K. Recall that h_K^+ is the narrow class number of K. Thus we have that the set $\{\alpha_1^{Dh_K^+}, \alpha_2^{Dh_K^+}, \ldots, \alpha_n^{Dh_K^+}\}$ coincides with the set $\{q_{\lambda_0}^{b'_1 Dh_K^+}, q_{\lambda_0}^{b'_2 Dh_K^+}, \ldots, q_{\lambda_0}^{b'_n Dh_K^+}\}$ in $\bar{\mathbb{F}}_\ell$ by $(*)$. Now we assume $\ell > 2c_n \ell_0^{dDh_K^+ M}$. Then we have

$$\{Dh_K^+ w_1(V)/2, \ldots, Dh_K^+ w_n(V)/2\} = \{b'_1 Dh_K^+, \ldots, b'_n Dh_K^+\}$$

by Lemma 6.12. Our result comes from this equation. $\square$

Proofs of Theorem 6.7 and 6.8. We only prove Theorem 6.7 because we can prove Theorem 6.8 by the same way. Suppose that there exists an ℓ-adic Galois representation V which is contained in $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet_{\mathrm{cycl}}$ and take a residual representation $\bar{V}$ of V. If we assume one of the situations (a) and (b) given in Proposition 6.10, then $\bar{V}$ is of uniform tame inertia weight $e_\lambda w/2$ at any place λ of K above ℓ, and thus $e_\lambda w/2$ must be a rational integer. Moreover, by Theorem 2.24, it follows that the tame inertia weight $e_\lambda w/2$ is between 0 and $e_\lambda r$. However, if we assume any one of the conditions (a), (b), (c) and (d), then $e_\lambda w$ is odd for some λ or $e_\lambda w/2 > e_\lambda r$. This is a contradiction. The rest of the assertion related with (e) follows from the fact ([**CS**], Theorem 1) that the sum of all the tame inertia weights of V at λ must be divisible by e_λ. $\square$

Remark 6.13. To remove the special assumption "ℓ does not split in K" in Theorem 6.8 is impossible in general because there exists such an example, which is pointed out by Akio Tamagawa: Let E be an elliptic curves over K with complex multiplication over K by an imaginary quadratic field $F := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_K(E) \subset K$. Then E is potential everywhere good reduction and thus we may suppose E has everywhere good reduction over K. Put $F_\ell := \mathbb{Q}_\ell \otimes_{\mathbb{Q}} F$, which is a semisimple $\mathbb{Q}_\ell$-algebra. It is well-known that F_ℓ acts faithfully on the Tate-module $V_\ell(E)$ of E and thus $V_\ell(E)$ has a natural structure of 1-dimensional F_ℓ-vector space. If ℓ splits in F, the decomposition $F_\ell \simeq \mathbb{Q}_\ell \times \mathbb{Q}_\ell$ induces a decomposition of $V_\ell(E)$ as a sum of 1-dimensional G_K-stable ℓ-adic representations. For such odd prime ℓ, it is easy to check that $V_\ell(E)$ is an element of the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$, where $\bullet = (2, 2, 1, 1)$.

3. Applications

We give some applications of our results. We use same notation as in the previous section.

3.1. Rasmussen-Tamagawa Conjecture. We consider the semistable reduction case of Conjecture 5.2.

DEFINITION 6.14. (1) We denote by $\mathcal{A}(K,g,\ell)_{\mathrm{st}}$ the set of K-isomorphism classes of abelian varieties in $\mathcal{A}(K,g,\ell)$ with semistable reduction everywhere.
(2) We denote by $\mathcal{A}(K,g,\ell_0,\ell)_{\mathrm{st}}$ the set of K-isomorphism classes of abelian varieties A over K with semistable reduction everywhere, of dimension g, which satisfy the following condition: The abelian variety A has good reduction at some places of K above ℓ_0 and $A[\ell]$ has a filtration of G_K-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]$$

such that $\bar{V}_k$ has dimension k for each $1 \leq k \leq 2g$.

Clearly, we see $\mathcal{A}(K,g,\ell)_{\mathrm{st}} \subset \mathcal{A}(K,g,\ell_0,\ell)_{\mathrm{st}}$ since $\ell \neq \ell_0$. The set $\mathcal{A}(K,g,\ell)_{\mathrm{st}}$ is finite, however, the set $\mathcal{A}(K,g,\ell_0,\ell)_{\mathrm{st}}$ may be infinite. The Rasmussen-Tamagawa Conjecture implies that $\mathcal{A}(K,g,\ell)_{\mathrm{st}}$ will be empty for a prime ℓ large enough.

Take an abelian variety A which is in the set $\mathcal{A}(K,g,\ell)_{\mathrm{st}}$ (resp. $\mathcal{A}(K,g,\ell_0,\ell)_{\mathrm{st}}$). Then the Tate module $V_\ell(A)$ of A is an element of the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}_{\mathrm{cycl}}$ (resp. $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^{\bullet}$) with $\bullet = (2g,2,1,1)$ (resp. $\bullet = (2g,\ell_0,1,1)$) for any $\ell > 2$ (resp. $\ell > \ell_0$). Consequently, we obtain the following results as corollaries of Theorem 6.7 and 6.8:

COROLLARY 6.15. *The set $\mathcal{A}(K,g,\ell)_{\mathrm{st}}$ is empty under any one of the following situations:*

(a) *$\ell \nmid d_K$ and $\ell > 2^{\delta_1}\binom{2g}{g}$, where $\delta_1 := 2dg+1$;*

(b) *The extension $K/\mathbb{Q}$ has odd degree and $\ell > 2^{\delta_2}\binom{2g}{g}$, where $\delta_2 := 2d^2g+1$.*

COROLLARY 6.16. *Suppose that ℓ does not split in K. The set $\mathcal{A}(K,g,\ell_0,\ell)_{\mathrm{st}}$ is empty under any one of the following situations:*

(a) *$\ell \nmid d_K$ and $\ell > 2\ell_0^{\delta'_1}\binom{2g}{g}$, where $\delta'_1 := 2dgh_K^+$;*

(b) *The extension $K/\mathbb{Q}$ has odd degree and $\ell > 2\ell_0^{\delta'_2}\binom{2g}{g}$, where $\delta'_2 := 2d^2gh_K^+$.*

REMARK 6.17. Rasmussen and Tamagawa have shown the finiteness of the set $\mathcal{A}(K,g)_{\mathrm{st}}$ by using the result of [**Ra**] instead of Theorem 2.24 (unpublished). Our main results in this paper are motivated by their work.

3.2. ℓ-torsion points of elliptic curves. We consider the following classical question:

QUESTION 6.18. *Does there exist a constant c_K, which depends only on K, such that for any semistable elliptic curve E defined over K without complex multiplication over K, the representation in its ℓ-torsion* *points $E[\ell]$ is irreducible whenever $\ell > c_K$? Furthermore, if the answer is positive, how can we evaluate such a constant c_K?*

By Mazur's results on a moduli of rational points of modular curve $X_0(N)$ ([**Ma**]), it is known that $c_{\mathbb{Q}} = 7$. If K is a quadratic field, then the existence of c_K is known and moreover, if the class number of K is 1, then the explicit calculation of c_K is given by Kraus [**Kr1**]. By combining results on Merel ([**Me**]) and Momose ([**Mo**]), Kraus showed the existence of c_K for a number field K which does not contain an imaginary quadratic field of class number 1 ([**Kr2**]). Moreover, Kraus defined the good condition "(C)" associated with K in *op. cit,* such that the existence and the explicit value of c_K is known if K satisfies this condition.

The following is easy consequence of Corollary 6.16 under the case $g = 1$.

Corollary 6.19. *Let E be an elliptic curve over K with everywhere semistable reduction. Let ℓ_E be the minimal prime number p such that E has good reduction at some finite places of K above p. Suppose ℓ does not split in K. Then $E[\ell]$ is irreducible under any one of the following conditions:*

(a) *$\ell \nmid d_K$ and $\ell > 4\ell_E^{\delta_1''}$, where $\delta_1'' := 2dh_K^+$;*
(b) *The extension $K/\mathbb{Q}$ has odd degree and $\ell > 4\ell_E^{\delta_2''}$, where $\delta_2'' := 2d^2h_K^+$.*

We remark that the above corollary is valid even if E has complex multiplication over K.

3.3. étale cohomology groups. For any semistable elliptic curve E over $\mathbb{Q}$, Serre proved the following ([**Se4**], Section 5.4, Proposition 21, Corollary 1): Let ℓ_E be the minimal prime number p such that E has good reduction at p. Then $E[\ell]$ is irreducible if $\ell > (1 + \ell_E^{1/2})^2$.

As a corollary of Theorem 6.8, we can slightly generalize this fact to étale cohomology groups of odd degree.

Corollary 6.20. *Let X be a proper smooth scheme over K with semistable reduction everywhere and w an odd integer. Let $b_w(X)$ be a w-th Betti number of X and ℓ_X the minimal prime number p such that X has good reduction at some places of K above p. Then there exists a constant C depending only on $b_w(X)$ and ℓ_X such that for any prime number $\ell > C$ which does not split in K, the étale cohomology group $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is not of residually Borel. More precisely, if ℓ does not split in K, $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is not of residually Borel under any one of the following conditions:*

(a) *$\ell \nmid d_K$ and $\ell > 2c_{b_w(X)}\ell_X^{\Delta_1}$, where $\Delta_1 := b_w(X)dh_K^+w$;*
(b) *The extension $K/\mathbb{Q}$ has odd degree and $\ell > 2c_{b_w(X)}\ell_X^{\Delta_2}$, where $\Delta_2 := b_w(X)d^2h_K^+w$.*

Proof. Putting $\bullet := (b_w(X), \ell_X, w, w)$, we see that the dual of $H^w_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is contained in the set $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$. Applying Theorem 6.8, we obtain the desired result. $\square$

For any proper smooth scheme X over K, there exists a finite extension L over K such that $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)$ is semistable everywhere as a representation of G_L for almost all ℓ by Proposition 6.3. For this L, we see that $H^w_{\text{ét}}(X_{\bar{L}}, \mathbb{Q}_\ell)^\vee$ satisfies (G-1), (G-2) and (G-3) as a representation of G_L. Thus if we can obtain the explicit description of L, it seems to be natural to hope that we may be able to obtain the analogous result of Corollary 6.20 for an arbitrary proper smooth scheme X (which may not have semistable reduction everywhere). However, there are some problems for this consideration. For example,

(a) it is very difficult to determine such an L in general.

(b) in general, a number field L which is a Galois extension of $\mathbb{Q}$ has infinitely many prime numbers which are non-split in L if and only if L is a cyclic extension of $\mathbb{Q}$ (by Chebotarev's density theorem). Hence Theorems 6.7 and 6.8 are not effective for representations of a non-cyclic Galois extension L of $\mathbb{Q}$.

For an abelian variety X over K, Raynaud's criterion of semistable reduction ([**Gr**], Proposition 4.7) implies that X has semistable reduction everywhere over $L := K(X[3], X[5])$. However this L may not be a cyclic extension of $\mathbb{Q}$ in general.

CHAPTER 7

Abelian case of the Rasmussen-Tamagawa Conjecture

In this chapter, we study the abelian case of the Rasmussen-Tamagawa Conjecture (Conjecture 5.2). We use the same notation as in Chapter 6. Denote by $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ the set of K-isomorphism classes of abelian varieties A in $\mathcal{A}(K,g,\ell)$ which satisfy the condition that $K(A[\ell^\infty])$ is an abelian extension of K. In fact, we define possibly an infinite set $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ in Section 2 which contains $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$, and prove that it is empty for any prime ℓ large enough.

THEOREM 7.1. *The set $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ is empty for any prime ℓ large enough.*

In particular, the set $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ is also empty for any prime ℓ large enough. The key to the proof of the above theorem is to construct a compatible system of Galois representations with a strong condition by using the Weil Conjecture, Faltings' trick in his proof of the Shafarevich Conjecture and Raynaud's criterion of semistable reduction.

1. Structures of compatible systems

Let E be a finite extension of $\mathbb{Q}$. For a finite place λ of E, we denote by ℓ_λ the prime number below λ, E_λ the completion of E at λ and $\mathbb{F}_\lambda$ the residue field of λ. Choose an algebraic closure $\bar{\mathbb{F}}_\lambda$ of $\mathbb{F}_\lambda$. Put $\chi_\lambda\colon G_K \xrightarrow{\chi_{\ell_\lambda}} \mathbb{Z}_{\ell_\lambda}^\times \hookrightarrow E_\lambda^\times$ and $\bar{\chi}_\lambda\colon G_K \xrightarrow{\bar{\chi}_{\ell_\lambda}} \mathbb{F}_{\ell_\lambda}^\times \hookrightarrow \mathbb{F}_\lambda^\times$, where χ_{ℓ_λ} and $\bar{\chi}_{\ell_\lambda}$ are the ℓ_λ-adic cyclotomic character and the mod ℓ_λ cyclotomic character, respectively. For a representation $\bar{\rho}_\lambda\colon G_K \to GL_n(\mathbb{F}_\lambda)$ with abelian semisimplification, Schur's lemma shows that $(\bar{\rho}_\lambda)^{\mathrm{ss}} \otimes \bar{\mathbb{F}}_\lambda$ conjugates to the direct sum of n characters, where the subscript "ss" means the semisimplification, and we call these n characters *characters associated with* $\bar{\rho}_\lambda$. For a λ-adic representation ρ_λ, we denote by $\bar{\rho}_\lambda$ a residual representation of ρ_λ (for a chosen lattice). Note that the isomorphism class of $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ is independent of the choice of a lattice by the Brauer-Nesbitt theorem.

Recall that we always suppress the notion of "defect set" and "ramification set" of compatible systems (Section 4 of Chapter 1).

THEOREM 7.2. *Let $(\rho_\lambda)_\lambda$ be an E-rational strictly compatible system of n-dimensional geometric semisimple λ-adic representations of G_K.*

Suppose that there exists an infinite set Λ *of finite places of* E *which satisfies the following:*

(1) *For any* $\lambda \in \Lambda$*, there exists a place* v *of* K *above* ℓ_λ *such that*
  (a) ρ_λ *is semistable at* v*.*
  (b) *there exist integers* $w_1 \leq w_2$ *which are independent of the choice of* $\lambda \in \Lambda$ *such that the Hodge-Tate weights of* $\rho_\lambda|_{G_v}$ *are in* $[w_1, w_2]$*.*

(2) *For any* $\lambda \in \Lambda$*,* $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ *is abelian and any character associated with* $\bar{\rho}_\lambda$ *has the form* $\varepsilon\bar{\chi}_\lambda^a$*, where* a *is an integer and* $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\lambda^\times$ *is a character unramified at all places of* K *above* ℓ_λ*.*

(3) *The Artin conductor of* $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ *is bounded independently of the choice of* $\lambda \in \Lambda$*.*

Then there exist integers $m_1, m_2, \ldots, m_n$ *and a finite extension* L *of* K *such that, for any* λ*, the representation* ρ_λ *is isomorphic to* $\chi_\lambda^{m_1} \oplus \chi_\lambda^{m_2} \oplus \cdots \oplus \chi_\lambda^{m_n}$ *on* G_L*.*

Proof. By replacing Λ with its infinite subset, we may suppose that ℓ_λ does not divide the discriminant of K and $\ell_\lambda > n$ for any $\lambda \in \Lambda$. Furthermore, we may assume that, for any $\lambda \in \Lambda$ and a finite place v of K above ℓ_λ as in the condition (1), the Hodge-Tate weights of $\rho_\lambda|_{G_v}$ viewed as a $\mathbb{Q}_{\ell_\lambda}$-representation are positive and bounded independently of the choice of $\lambda \in \Lambda$. By the condition (3), there exists an ideal $\mathfrak{n}$ of $\mathcal{O}_K$ such that, for any $\lambda \in \Lambda$, the Artin conductor outside ℓ_λ of $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ divides $\mathfrak{n}$. If we denote by ψ a character associated with $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ for $\lambda \in \Lambda$ and decompose $\psi = \varepsilon\bar{\chi}^a$ where ε is as in the condition (2), then the Artin conductor outside ℓ_λ of ε also divides $\mathfrak{n}$. Hence, replacing the field K with the strict ray class field of K associated with $\mathfrak{n}$, we may replace the condition (2) with the following condition (2)$'$:

(2)$'$ For any $\lambda \in \Lambda$, $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ is abelian and any character associated with $\bar{\rho}_\lambda$ has the form $\bar{\chi}_\lambda^a$.

Now take any $\lambda \in \Lambda$. Let $\bar{\chi}_\lambda^{a_{\lambda,1}}, \bar{\chi}_\lambda^{a_{\lambda,2}}, \ldots, \bar{\chi}_\lambda^{a_{\lambda,n}}$ be all the characters associated with $\bar{\rho}_\lambda$. By the condition (2)$'$ and $\ell_\lambda > n$, the representation $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ conjugates to the direct some of n characters (over $\mathbb{F}_\lambda$) of the form $\bar{\chi}_\lambda^a$, which has values in $\mathbb{F}_{\ell_\lambda}^\times$. Hence if we regard the $\mathbb{F}_\lambda$-representation $\bar{\rho}_\lambda$ as an $\mathbb{F}_{\ell_\lambda}$-representation, its semisimplification is of a diagonal form whose diagonal components are the copies of $\bar{\chi}_{\ell_\lambda}^{a_{\lambda,1}}, \bar{\chi}_{\ell_\lambda}^{a_{\lambda,2}}, \ldots, \bar{\chi}_{\ell_\lambda}^{a_{\lambda,n}}$. Furthermore, it is a direct summand of the semisimplification of a residual representation of ρ_λ viewed as a $\mathbb{Q}_{\ell_\lambda}$-representation. Therefore, by Caruso's result on an upper bound for tame inertia weights (cf. Theorem 2.24) and the condition (1), there exists a constant $C > 0$, which is independent of the choice of $\lambda \in \Lambda$, and an integer $0 \leq b_{\lambda,i} \leq C$ such that

$$(\sharp) \quad b_{\lambda,i} \equiv a_{\lambda,i} \bmod \ell_\lambda - 1$$

for any i (recall that ℓ_λ does not divide the discriminant of K). Now we claim that the set $\{b_{\lambda,1}, b_{\lambda,2}, \ldots, b_{\lambda,n}\}$ is independent of the choice of $\lambda \in \Lambda$ large enough. Denote by S the ramification set of $(\rho_\lambda)_\lambda$. Take a $v_0 \notin S$ and decompose $\det(T - \rho_\lambda(\mathrm{Fr}_{v_0})) = \prod_{j=1}^n (T - \alpha_{v_0,j})$. By conditions (2)' and ($\sharp$), we have the congruence $\prod_{j=1}^n (T - \alpha_{v_0,j}) \equiv \prod_{j=1}^n (T - q_{v_0}^{b_{\lambda,j}})$ in $\bar{\mathbb{F}}_\lambda[T]$. If ℓ_λ is large enough (note that Λ is an infinite set), then we obtain that this congruence is in fact an equality in $E[T]$: $\prod_{j=1}^n (T - \alpha_{v_0,j}) = \prod_{j=1}^n (T - q_{v_0}^{b_{\lambda,j}})$. Therefore, the set $\{b_{\lambda,1}, b_{\lambda,2}, \ldots, b_{\lambda,n}\}$ is independent of the choice of $\lambda \in \Lambda$ with ℓ_λ large enough. This proves the claim. We denote $\{b_{\lambda,1}, b_{\lambda,2}, \ldots, b_{\lambda,n}\}$ by $\{m_1, m_2, \ldots, m_n\}$ for such a $\lambda \in \Lambda$. By the compatibility of $(\rho_\lambda)_\lambda$, we obtain the equation $\det(T - \rho_\lambda(\mathrm{Fr}_v)) = \prod_{j=1}^n (T - q_v^{m_j})$ for any λ and $v \notin S_{\ell_\lambda}$ (recall $S_{\ell_\lambda} := S \cup \{\text{places of } K \text{ above } \ell_\lambda\}$, Section 4 of Chapter 1). Therefore, the representation ρ_λ is isomorphic to $\chi_\lambda^{m_1} \oplus \chi_\lambda^{m_2} \oplus \cdots \oplus \chi_\lambda^{m_n}$. By the compatibility of $(\rho_\lambda)_\lambda$, this finishes the proof. $\square$

Corollary 7.3. *Let $(\bar{\rho}_\lambda)_\lambda$ be an E-rational strictly compatible system of abelian semisimple mod λ representations of G_K. Suppose that, for infinitely many finite places λ of E, any character associated with $\bar{\rho}_\lambda$ has the form $\varepsilon\bar{\chi}_\lambda^a$, where $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\lambda^\times$ is a character unramified at all places of K above ℓ_λ. Then there exist a finite extension L of K and integers $m_1, m_2, \ldots, m_n$ such that, for all finite places λ of E, the representation $\bar{\rho}_\lambda$ is isomorphic to $\bar{\chi}_\lambda^{m_1} \oplus \bar{\chi}_\lambda^{m_2} \oplus \cdots \oplus \bar{\chi}_\lambda^{m_n}$ on G_L.*

Proof. By Theorem 1.21, we know that there exist a finite extension E' of E and an E'-rational abelian semisimple compatible system $(\rho_{\lambda'})_{\lambda'}$ of λ'-adic representations of G_K which arises from Hecke characters such that $(\rho_{\lambda'})_{\lambda'}$ is a lift of $(\bar{\rho}_\lambda)_\lambda$, that is, $\bar{\rho}_{\lambda'}$ is isomorphic to $\bar{\rho}_\lambda \otimes \mathbb{F}_{\lambda'}$ for any λ and any finite place λ' of E' above λ. By Corollary 1.22, the compatible system $(\rho_{\lambda'})_{\lambda'}$ satisfies all the assumptions (1), (2) and (3) in Theorem 7.2, and consequently we obtain the desired result. $\square$

Corollary 7.4. *Let $(\rho_\lambda)_\lambda$ be an E-rational strictly compatible system of n-dimensional semisimple λ-adic representations of G_K. Suppose that*

- (i) *$(\bar{\rho}_\lambda)^{\mathrm{ss}}$ is abelian for almost all λ;*
- (ii) *for infinitely many λ, any character associated with $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ has the form $\varepsilon\bar{\chi}_\lambda^a$, where $\varepsilon\colon G_K \to \mathbb{F}_\lambda^\times$ is a character unramified at all places of K above ℓ_λ.*

Then there exist integers $m_1, m_2, \ldots, m_n$ and a finite extension L of K such that, for any λ, the representation ρ_λ is isomorphic to $\chi_\lambda^{m_1} \oplus \chi_\lambda^{m_2} \oplus \cdots \oplus \chi_\lambda^{m_n}$ on G_L.

Proof. The result follows immediately by applying Corollary 7.3 to the compatible system $((\bar{\rho}_\lambda)^{\mathrm{ss}})_\lambda$. $\square$

Let λ and λ' be finite places of K of different residual characteristics. Let ρ_λ be an E-rational n-dimensional semisimple λ-adic representations of G_K with ramification set S. Suppose that there exists an semisimple λ'-adic representation $\rho_{\lambda'}$ of G_K such that

$$\det(T - \rho_\lambda(\mathrm{Fr}_v)) = \det(T - \rho_{\lambda'}(\mathrm{Fr}_v))$$

for any $v \notin S_{\ell_\lambda} \cup S_{\ell_{\lambda'}}$. Considering Fontaine-Mazur's "Main Conjecture" (Conjecture 1.10), we hope that $\rho_{\lambda'}$ is crystalline for any finite place v' of K above $\ell_{\lambda'}$[1]. However to prove this hope seems not to be easy. If ρ_λ is abelian, the hope is true by Theorem 1.16 and 1.17. If we consider representations which is pure (cf. Chapter 1, Section 4), we can improve the statement (1) of Theorem 7.2 as below. If the hope is true, it is not difficult to prove the proposition below without the assumption of pureness by the similar method of the proof of Theorem 7.2.

PROPOSITION 7.5. *Let $(\rho_\lambda)_\lambda$ be an E-rational strictly compatible system of n-dimensional geometric semisimple λ-adic representations of G_K. Suppose that $(\rho_\lambda)_\lambda$ is pure. Suppose that there exists an infinite set Λ of finite places of K which satisfies the following:*

(1) *For any $\lambda \in \Lambda$, there exists a place v of K above ℓ_λ such that*
   (a) *there exists a constant $C > 0$ which is independent of the choice of $\lambda \in \Lambda$ such that $[I_v : \mathfrak{L}_v(\rho_\lambda)] < C$. Here $\mathfrak{L}_v(\rho_\lambda)$ is the inertial level of ρ_λ at v (see Section 2.4).*
   (b) *there exist integers $w_1 \le w_2$ which are independent of the choice of $\lambda \in \Lambda$ such that the Hodge-Tate weights of $\rho_\lambda|_{G_v}$ are in $[w_1, w_2]$.*

(2) *For any $\lambda \in \Lambda$, $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ is abelian and any character associated with $\bar{\rho}_\lambda$ has the form $\varepsilon\bar{\chi}_\lambda^a$, where $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\lambda^\times$ is a character unramified at all places of K above ℓ_λ.*

(3) *For any $\lambda \in \Lambda$, the Artin conductor of $(\bar{\rho}_\lambda)^{\mathrm{ss}}$ is bounded independently of the choice of $\lambda \in \Lambda$.*

Then there exist an integer m and a finite extension L of K such that, for any λ, the representation ρ_λ is isomorphic to $(\chi_\lambda^m)^{\oplus n}$ on G_L.

PROOF. Most parts of the first paragraph of this proof will proceed by the similar method as the proof of Theorem 7.2 and hence we will often omit precise arguments. First we may assume that, for any $\lambda \in \Lambda$,

(2)$'$ any character associated with $\bar{\rho}_\lambda$ has the form $\bar{\chi}_\lambda^a$

and furthermore, $\rho_\lambda|_{G_v}$ has Hodge-Tate weights in $[0, r]$ for any λ and v as in the condition (1). Here r is a positive integer which is independent of the choice of $\lambda \in \Lambda$. Suppose λ is a finite place in Λ. Let $\bar{\chi}_\lambda^{a_{\lambda,1}}, \bar{\chi}_\lambda^{a_{\lambda,2}}, \dots, \bar{\chi}_\lambda^{a_{\lambda,n}}$ be all the characters associated with $\bar{\rho}_\lambda$. Taking a

[1]Because ρ_λ and $\rho_{\lambda'}$ shall come from an algebraic variety X and their ramification set S shall be "bad primes" of X.

finite place v as in the condition (1), there exists a finite extension L_w of K_v such that $\rho_\lambda|_{G_{L_w}}$ is semistable and $[L_w : K_v] \leq C$. If we denote by e_w the absolute ramification index of L_w, then it follows $e_w \leq C[K : \mathbb{Q}]$ and Theorem 2.24 implies that there exists an integer $0 \leq b'_{\lambda,i} \leq e_w r$ which satisfies $b'_{\lambda,i} \equiv e_w a_{\lambda,i} \bmod \ell_\lambda - 1$. Consequently, we see that there exist integers $e > 0$ and $D > 0$, which are independent of the choice of $\lambda \in \Lambda$ and $b_{\lambda,i} \equiv e a_{\lambda,i} \bmod \ell_\lambda - 1$ for some integer $b_{\lambda,i} \in [0, D]$. Take any $v \notin S_{\ell_\lambda}$ and decompose $\det(T - \rho_\lambda(\mathrm{Fr}_v)) = \prod_{j=1}^n (T - \alpha_{v,j})$. Then, by the similar arguments as the proof of Theorem 7.2, we can show that $\prod_{j=1}^n (T - \alpha_{v,j}^e) = \prod_{j=1}^n (T - q_v^{b_{\lambda,j}})$ if we take $\lambda \in \Lambda$ with ℓ_λ large enough. Since $(\rho_\lambda)_\lambda$ is pure, we have

$$\prod_{j=1}^n (T - \alpha_{v,j}^e) = \prod_{j=1}^n (T - q_v^b)$$

for some integer b. It follows from the compatibility of $(\rho_\lambda)_\lambda$ that the above equation holds for any λ (which may not be in Λ) and $v \notin S_{\ell_\lambda}$.

In the argument below, we use the method of the proof of Proposition 1.2 of [**KL**]. Fix λ and denote it by λ_0. Take a finite extension K' of K such that there exists a continuous character $\chi_{\lambda_0}^{1/e} \colon G_{K'} \to E_{\lambda_0}^\times$ which has values in the integer ring of E_{λ_0} and $(\chi_{\lambda_0}^{1/e})^e = \chi_{\lambda_0}$. Replace this K' with K. Then we know that, for any $v \notin S_{\ell_{\lambda_0}}$, all the roots of $\det(T - \rho'_{\lambda_0}(\mathrm{Fr}_v))$ are roots of unity, where ρ'_{λ_0} is the twist of ρ_{λ_0} by $(\chi_{\lambda_0}^{1/e})^{-b}$. Since there are only finitely many such roots of unity, there are only finitely many possibilities for the characteristic polynomial of Fr_v. Hence the function which takes $g \in G_K$ to $\det(T - \rho'_{\lambda_0}(g)) \in E[T]$ is continuous and takes only finitely many values by Chebotarev's density theorem. It follows that the set $\{g \in G_K \mid \det(T - \rho'_{\lambda_0}(g)) = (T-1)^n\}$ is an open subset of G_K, which contains the identity map of $\bar{K}$. Hence there exists a finite extension L of K such that $G_L \subset \{g \in G_K \mid \det(T - \rho'_{\lambda_0}(g)) = (T-1)^n\}$. Then we see that ρ_{λ_0} is isomorphic to $((\chi_{\lambda_0}^{1/e})^b)^{\oplus n}$ on G_L. Since ρ_{λ_0} is geometric, we know that $b/e =: m$ is an integer and we finish the proof by the compatibility of $(\rho_\lambda)_\lambda$. □

2. Notion and the proof of Theorem 7.1

2.1. Notion. We give precise definitions for the statement of Theorem 7.1. Let K be a finite extension of $\mathbb{Q}$ and A an abelian variety over K. Consider the following conditions:

- (RT_ℓ) $K(A[\ell])$ is an ℓ-extension of $K(\mu_\ell)$.
- $(\mathrm{RT}_\ell)'$ For some finite extension L of K which is unramified at all places above ℓ, $L(A[\ell])$ is an ℓ-extension of $L(\mu_\ell)$.
- $(\mathrm{RT}_{\mathrm{red}})$ The abelian variety A has good reduction away from ℓ over K.
- $(\mathrm{RT}_{\mathrm{ab}})$ $K(A[\ell^\infty])$ is an abelian extension of K.

It is clear that (RT_ℓ) implies (RT_ℓ)$'$. Note that, for a g-dimensional abelian variety A over K, the isomorphism class $[A]$ of A is contained in $\mathcal{A}(K,g,\ell)$ if and only if A satisfies (RT_ℓ) and ($\mathrm{RT}_{\mathrm{red}}$) by the criterion of Neron-Ogg-Shafarevich.

Definition 7.6. We define sets $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ and $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ of isomorphism classes of g-dimensional abelian varieties A over K as follows:
(1) $[A] \in \mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ if and only if A satisfies (RT_ℓ), ($\mathrm{RT}_{\mathrm{red}}$) and ($\mathrm{RT}_{\mathrm{ab}}$).
(2) $[A] \in \mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ if and only if A satisfies (RT_ℓ)$'$ and ($\mathrm{RT}_{\mathrm{ab}}$).

Clearly, we have $\mathcal{A}(K,g,\ell) \supset \mathcal{A}(K,g,\ell)_{\mathrm{ab}} \subset \mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$. The set $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ is always finite but $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ may be infinite[2]. Here is a table of definitions and properties for cardinality about our sets defined as above.

	(RT_ℓ)	(RT_ℓ)$'$	($\mathrm{RT}_{\mathrm{red}}$)	($\mathrm{RT}_{\mathrm{ab}}$)	cardinality
$\mathcal{A}(K,g,\ell)$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$-$	finite
$\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$	$\bigcirc$	$\bigcirc$	$\bigcirc$	$\bigcirc$	finite
$\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$	$-$	$\bigcirc$	$-$	$\bigcirc$	$-$

Here, the meaning of the notations are as follows: Every K-isomorphism class of A satisfies (RT_*) (or (RT_ℓ)$'$) in the cases marked with $\bigcirc$. A "finite" means that there exist only finitely many such isomorphism class of abelian varieties in that case.

If we admit Conjecture 5.2, then we know that $\mathcal{A}(K,g,\ell)_{\mathrm{ab}}$ is empty for any prime ℓ large enough. In fact, we can show that $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ is empty for any prime ℓ large enough.

2.2. Proof of Theorem 7.1. First we study the structure of $A[\ell]$ for an abelian variety A in $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$. Let A be any g-dimensional abelian variety over K. We denote by $\rho_{A,\ell}\colon G_K \to GL_{2g}(\mathbb{Q}_\ell)$ (resp. $\bar{\rho}_{A,\ell}\colon G_K \to GL_{2g}(\mathbb{F}_\ell)$) the representation attached to the ℓ-adic Tate module $T_\ell(A)$ of A (resp. the set $A[\ell]$ of ℓ-torsion points of A). Consider the following properties:

($\mathrm{RT}_{\mathrm{st}}$) $(\bar{\rho}_{A,\ell})^{\mathrm{ss}}$ conjugates to the direct sum of n characters which are of the form $\bar{\chi}_\ell^a$.

($\mathrm{RT}_{\mathrm{st}}$)$'$ $(\bar{\rho}_{A,\ell})^{\mathrm{ss}}$ is abelian and characters associated with $\bar{\rho}_{A,\ell}$ are of the form $\varepsilon\bar{\chi}_\ell^a$, where $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\ell^\times$ is a continuous character which is unramified at all places above ℓ.

Recall that the condition (RT_ℓ) is equivalent to the condition ($\mathrm{RT}_{\mathrm{st}}$) by Corollary 5.4. Hence the isomorphism class $[A]$ of g-dimensional abelian variety A over K is in $\mathcal{A}(K,g,\ell)$ if and only if A satisfies the condition ($\mathrm{RT}_{\mathrm{red}}$) and ($\mathrm{RT}_{\mathrm{st}}$).

[2]The author does not know an example such that $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ is infinite.

LEMMA 7.7. *Let A be a g-dimensional abelian variety over K. Suppose that the abelian variety A satisfies the condition* ($\mathrm{RT}_{\mathrm{ab}}$)*. Then A satisfies the condition* $(\mathrm{RT}_\ell)'$ *if and only if A satisfies the condition* $(\mathrm{RT}_{\mathrm{st}})'$*.*

PROOF. Suppose that an abelian variety A satisfies the condition ($\mathrm{RT}_{\mathrm{ab}}$) and denote by $\psi_1, \ldots, \psi_{2g}$ characters associated with $\bar{\rho}_{A,\ell}$. If A satisfies $(\mathrm{RT}_{\mathrm{st}})'$, then we have $\psi_i = \varepsilon_i \bar{\chi}_\ell^{a_i}$ for some integer a_i where $\varepsilon_i \colon G_K \to \bar{\mathbb{F}}_\ell^\times$ is a continuous character which is unramified at all places of K above ℓ. Let L be the composition field of all fields $\bar{K}^{\ker \varepsilon_i}$ for all i. Then L is unramified at all places of K above ℓ. Since each $\psi_i|_{G_{L(\mu_\ell)}}$ is trivial, we obtain $(\mathrm{RT}_\ell)'$. Conversely, suppose that $(\mathrm{RT}_\ell)'$ holds and take a field L as in $(\mathrm{RT}_\ell)'$. By Lemma 5.3, we know that each $\psi_i|_{G_L}$ is equal to $\bar{\chi}_\ell^{a_i}$ for some integer a_i. Hence $\varepsilon_i := \psi_i \cdot \bar{\chi}_\ell^{-a_i} \colon G_K \to \bar{\mathbb{F}}_\ell^\times$ is unramified at all places above ℓ and this implies $(\mathrm{RT}_{\mathrm{st}})'$. □

We recall the following proposition, which is used in Faltings' proof of the Shafarevich Conjecture [**Fa**].

PROPOSITION 7.8. *There exists a finite set T of finite places of K, depending only on K, E, S, n and λ, which satisfies the following property:*

(1) *The intersection $T \cap S_{\ell_\lambda}$ is empty.*
(2) *If ρ and ρ' are semisimple λ-adic representations $G_K \to GL_n(E_\lambda)$ unramified outside S with $\mathrm{Tr}(\rho(\mathrm{Fr}_v)) = \mathrm{Tr}(\rho'(\mathrm{Fr}_v))$ for any $v \in T$, then ρ and ρ' are isomorphic.*

PROOF. We give a proof only in the case where $E = \mathbb{Q}$ for simplicity (in general case, the proof proceeds in the same way).

First we define the set T as below. By the theorem of Hermite-Minkowski, there exists a finite Galois extension L of K such that L contains all the finite extensions K' of K which are unramified outside S_ℓ with $[K' : K] \leq \ell^{2n^2}$. By the Chebotarev density theorem, we can choose a finite set T such that all the images of arithmetic Frobenii for $v \in T$ in $\mathrm{Gal}(L/K)$ generates $\mathrm{Gal}(L/K)$. Let $\rho_1 \colon G_K \to GL_1(V_1) \simeq GL_n(\mathbb{Q}_\ell)$ and $\rho_1 \colon G_K \to GL_2(V_2) \simeq GL_n(\mathbb{Q}_\ell)$ be representations such that

$$\mathrm{Tr}\rho_1(\mathrm{Fr}_v) = \mathrm{Tr}\rho_2(\mathrm{Fr}_v) \quad \text{for any } v \in T.$$

Take a G_K-stable lattice T_i of V_i for each i. Put $\rho_{12} := \rho_1 \times \rho_2 \colon G_K \to GL(T_1) \times GL(T_2)$. Let M be the sub $\mathbb{Z}_\ell$-module of $\mathrm{End}_{\mathbb{Z}_\ell}(T_1) \times \mathrm{End}_{\mathbb{Z}_\ell}(T_2)$ which is generated by $\mathrm{Im}(\rho_{12})$, that is, $M := \mathbb{Z}_\ell[\mathrm{Im}(\rho_{12})]$. Then M has a natural action on T_i via the projection $M \to \mathrm{End}_{\mathbb{Z}_\ell}(T_i)$. Since $\mathrm{End}_{\mathbb{Z}_\ell}(T_1) \times \mathrm{End}_{\mathbb{Z}_\ell}(T_2)$ has rank $2n^2$, we have that M is a free $\mathbb{Z}_\ell$-module of rank $\leq 2n^2$. Our goal is to show that

$$(*) \quad \mathrm{Tr}(m \mid T_1) = \mathrm{Tr}(m \mid T_2) \quad \text{for any } m \in M,$$

and it is enough to prove that the above equality $(*)$ holds for generators of M. Since the equality $(*)$ holds for any $v \in T$ and $m = \rho_{12}(\mathrm{Fr}_v)$, we finish the proof if we prove that $\{\rho_{12}(\mathrm{Fr}_v) \mid v \in T\}$ generates M as a $\mathbb{Z}_\ell$-module. Denote by N the sub $\mathbb{Z}_\ell$-module of M which is generated by $\{\rho_{12}(\mathrm{Fr}_v) \mid v \in T\}$. It is enough to show that the image of N in $M/\ell M$ generates $M/\ell M$ by Nakayama's lemma. Put $\bar{\rho}_{12}\colon G_K \stackrel{\rho_{12}}{\to} M \stackrel{\mathrm{Proj}}{\longrightarrow} M/\ell M$. Note that $\bar{\rho}_{12}$ has values in $(M/\ell M)^\times$ and hence we have a homomorphism $\psi\colon G_K \to (M/\ell M)^\times$ induced by $\bar{\rho}_{12}$. Since ψ is unramified outside S_ℓ and $\mathrm{Im}(\bar{\rho}_{12}) \leq \ell^{\dim_{\mathbb{F}_\ell} M/\ell M} \leq \ell^{2n^2}$, we see that ψ factors through $\mathrm{Gal}(L/K)$. By this fact, we obtain $\mathrm{Im}(\bar{\rho}_{12}) = \{\bar{\rho}_{12}(\mathrm{Fr}_v) \mid v \in T\} \subset M/\ell M$ and therefore, we obtain

$$M/\ell M = \mathbb{F}_\ell[\mathrm{Im}(\bar{\rho}_{12})] = N.$$

This completes the proof. □

Corollary 7.9. *Fix an integer w. The set of isomorphism classes of semisimple n-dimensional ℓ-adic representations $G_K \to GL_n(\mathbb{Q}_\ell)$ which are $\mathbb{Q}$-integral with Frobenius weights $\leq w$ outside S, is finite.*

Proof. Take a finite set T which appears in Proposition 7.8 under the condition $E = \mathbb{Q}$. For any $v \in T$ and an ℓ-adic representation $G_K \to GL_n(\mathbb{Q}_\ell)$ which is $\mathbb{Q}$-integral with Frobenius weights $\leq w$ outside S, there are only finitely many possibilities for the trace $\mathrm{Tr}(\rho(\mathrm{Fr}_v))$ of Fr_v. Hence Proposition 7.8 implies the desired result. □

Proposition 7.10 (Raynaud's criterion of semistable reduction, [**Gr**], Proposition 4.7). *Suppose A is an abelian variety over a field F with a discrete valuation v, n is a positive integer not divisible by the residue characteristic, and the points of $A[n]$ are defined over an extension of F which is unramified over v. If $n \geq 3$ then A has semistable reduction at v. In particular, if A is an abelian variety over a number field K, then A has semistable reduction everywhere over $K(A[12]) = K(A[3], A[4])$.*

For an integer $g > 0$, put

$$\begin{aligned} D_g &:= \sharp GSp_{2g}(\mathbb{Z}/12\mathbb{Z}) \\ &= \sharp GSp_{2g}(\mathbb{Z}/3\mathbb{Z}) \cdot \sharp GSp_{2g}(\mathbb{Z}/4\mathbb{Z}). \end{aligned}$$

This integer plays an important role in the following proof.

Corollary 7.11. *Fix an integer $g > 0$. For any g-dimensional abelian variety A over K, there exists a finite Galois extension L of K such that $[L : K]$ divides D_g and A has semistable reduction everywhere over L.*

PROOF. Since $K(A[12])$ is a Galois extension of K and $[K(A[12]):K]$ divides[3] D_g, the desired assertion follows from the last sentence of Proposition 7.10. $\square$

If $\rho\colon G_K \to GL_{2g}(\mathbb{Q}_\ell)$ is an abelian representation, then, for any integer k, we denote by ρ^k the representation $G_K \to GL_{2g}(\mathbb{Q}_\ell)$ which is defined by $\rho^k(s) := (\rho(s))^k$ for any $s \in G_K$. With this notation, combining the above corollaries, we obtain the following lemma which plays an important role in the proof of Theorem 7.1 to construct a good compatible system.

LEMMA 7.12. *Let $g > 0$ be an integer and ℓ_0 a prime number. Let $\mathcal{A}_{\ell_0}$ be the set of isomorphism classes of representations $\rho\colon G_K \to GL_{2g}(\mathbb{Q}_{\ell_0})$ which are isomorphic to $\rho_{A,\ell_0}^{D_g}$ for some g-dimensional abelian variety A over K such that $K(A[\ell_0^\infty])$ is an abelian extension of K. Then $\mathcal{A}_{\ell_0}$ is finite.*

PROOF. If A is an abelian variety over K such that $K(A[\ell_0^\infty])$ is an abelian extension of K, then A has potential good reduction everywhere. Thus it follows from Corollary 7.11 that the representation $\rho_{A,\ell_0}^{D_g}$ is unramified outside ℓ_0 for any g-dimensional abelian variety A over K such that $K(A[\ell_0^\infty])$ is an abelian extension of K. Take any finite place v of K not above ℓ_0. Take a finite Galois extension L of K such that $[L:K]$ divides D_g and that A has good reduction everywhere over L. Let v_L be a finite place of L above v and denote by f the extension degree of $\mathbb{F}_{v_L}$ over $\mathbb{F}_v$, where $\mathbb{F}_{v_L}$ and $\mathbb{F}_v$ are residue fields of v_L and v, respectively. Noting that L is a Galois extension of K and A has good reduction everywhere over L, we see that D_g/f is an integer and obtain the equation

$$\det(T - \rho_{A,\ell_0}^{D_g}(\mathrm{Fr}_v)) = \det(T - (\rho_{A,\ell_0}(\mathrm{Fr}_{v_L}))^{D_g/f}).$$

Since A has good reduction everywhere over L, the Weil Conjecture implies that $\det(T - \rho_{A,\ell_0}(\mathrm{Fr}_{v_L}))$ has rational integer coefficients and hence so is $\det(T-(\rho_{A,\ell_0}(\mathrm{Fr}_{v_L}))^{D_g/f})$. Consequently, the representation $\rho_{A,\ell_0}^{D_g}$ is $\mathbb{Q}$-integral with Frobenius weight $D_g/2$ outside the set of finite places of K above ℓ_0. By Corollary 7.9, we have the desired result (note that $\rho_{A,\ell_0}^{D_g}$ is semisimple). $\square$

PROOF OF THEOREM 7.1. First we note that, if an abelian variety A over K satisfies $(\mathrm{RT}_{\mathrm{ab}})$, then $\rho_{A,\ell'}$ is abelian for any prime number ℓ' (cf. [**Se3**], Chapter III, Section 2.3, Corollary 1). Fix a prime number ℓ_0 and denote by $\mathcal{A}_{\ell_0}$ the set as in Lemma 7.12. Assume that there exist infinitely many prime numbers ℓ such that $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$ is not

[3]The image of the representation $\rho_{A,n}\colon G_K \to GL(A[n]) \simeq GL_{2g}(\mathbb{Z}/n\mathbb{Z})$ lands inside $GSp_{2g}(\mathbb{Z}/n\mathbb{Z}) \subset GL_{2g}(\mathbb{Z}/n\mathbb{Z})$ thanks to the Galois-equivariance of the Weil pairing.

empty. For every such ℓ, we obtain the ℓ_0-adic representation $\rho_{A,\ell_0}^{D_g}$ which is in the set $\mathcal{A}_{\ell_0}$, where A is an abelian variety whose isomorphism class is in the set $\mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$. By Lemma 7.12, we see that there exists a representation ρ_{ℓ_0} in $\mathcal{A}_{\ell_0}$ such that for infinitely many ℓ and $[A] \in \mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$, $\rho_{A,\ell_0}^{D_g}$ is isomorphic to ρ_{ℓ_0}. By the Weil Conjecture, the representation ρ_{ℓ_0} extends to a $\mathbb{Q}$-integral strict compatible system $(\rho_\ell)_\ell$ of $2g$-dimensional abelian semisimple ℓ-adic representations of G_K. Furthermore, for infinitely many prime numbers ℓ, the characters associated with $\bar{\rho}_\ell$ are of the form $\varepsilon\bar{\chi}_\ell^a$ by Lemma 7.7, where $\varepsilon\colon G_K \to \bar{\mathbb{F}}_\ell^\times$ is a continuous character which is unramified at all places of K above ℓ. Applying Theorem 7.2 (or Corollary 7.4), we see that there exist integers $m_1,\dots,m_{2g}$ and a finite extension L of K such that ρ_{ℓ_0} is isomorphic to $\chi_{\ell_0}^{m_1} \oplus \chi_{\ell_0}^{m_2} \oplus \cdots \oplus \chi_{\ell_0}^{m_{2g}}$ on G_L. In particular, for some prime number ℓ and $[A] \in \mathcal{A}'(K,g,\ell)_{\mathrm{ab}}$, $\rho_{A,\ell_0}^{D_g}$ is isomorphic to $\chi_{\ell_0}^{m_1} \oplus \chi_{\ell_0}^{m_2} \oplus \cdots \oplus \chi_{\ell_0}^{m_{2g}}$ on G_L. Therefore, looking at the eigenvalues of images of a Frobenius element (at some place) of $\rho_{A,\ell_0}^{D_g}$ and $\chi_{\ell_0}^{m_1} \oplus \chi_{\ell_0}^{m_2} \oplus \cdots \oplus \chi_{\ell_0}^{m_{2g}}$, we know that $D_g/2 = m_1 = m_2 = \cdots = m_{2g}$ by the Weil Conjecture. Since $\rho_{A,\ell_0}^{D_g}$ has Hodge-Tate weights 0 and D_g at a place of L above ℓ_0, this is a contradiction. $\square$

3. Application to CM abelian varieties

Theorem 7.1 gives some implication for abelian varieties which have complex multiplication. If an abelian variety A over K has complex multiplication over K, then it is well-known that $\rho_{A,\ell}$ is abelian (cf. [**ST**], Section 4, Corollary 2).

Lemma 7.13. *Let A be an abelian variety over K of dimension g. Suppose that there exists a finite extension L of K which satisfy the following two properties:*

(i) *$L \cap K(A[\ell_0^\infty]) = K$ for some prime ℓ_0; and*
(ii) *the abelian variety A has complex multiplication over L.*

Then $\rho_{A,\ell}$ is abelian for any prime ℓ.

Proof. The condition (i) implies that $\rho_{A,\ell}(G_K) = \rho_{A,\ell}(G_L)$, which is an abelian group by (ii). Thus ρ_{A,ℓ_0} is abelian and hence $\rho_{A,\ell}$ is abelian for any prime ℓ (cf. [**Se3**], Chapter III, Section 2.3, Corollary 1). $\square$

For example, the condition (i) is automatically satisfied if $[L:K]$ is prime to $\sharp GL_{2g}(\mathbb{F}_{\ell_0}) = (\ell_0^{2g}-1)(\ell_0^{2g}-\ell)\cdots(\ell_0^{2g}-\ell_0^{2g-1})$. Combining Lemma 7.13 with Theorem 7.1, we obtain the following.

Corollary 7.14. *The set of isomorphism classes $[A] \in \mathcal{A}(K,g,\ell)$ of abelian varieties A as in Lemma 7.13 is empty for any prime ℓ large enough.*

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