A note on highly Kummer-faithful fields

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Abstract

We introduce a notion of *highly Kummer-faithful fields* and study its relationship with the notion of Kummer-faithful fields. We also give some examples of highly Kummer-faithful fields. For example, if k is a number field of finite degree over $\mathbb{Q}$, g is an integer > 0 and $\mathbf{m} = (m_p)_p$ is a family of non-negative integers, where p ranges over all prime numbers, then the extension field $k_{g,\mathbf{m}}$ obtained by adjoining to k all coordinates of the elements of the p^{m_p}-torsion subgroup $A[p^{m_p}]$ of A for all semi-abelian varieties A over k of dimension at most g and all prime numbers p, is highly Kummer-faithful.

1 Introduction

The notion of a Kummer-faithful field, defined by Mochizuki [Mo], plays an important role in anabelian geometry; it is defined by the non-divisibility of the Mordell-Weil groups of semi-abelian varieties over the field (see Definition 2.2), so that Kummer theory works effectively for the purpose of reconstructing geometric objects over such a field. Recently, some versions of the Grothendieck Conjecture in anabeian geometry are being generalized from ones over sub-p-adic fields to ones over Kummer-faithful fields (see, e.g., [Ho]). Note that a sub-p-adic field (for a prime number p) is Kummer-faithful ([Mo, Remark 1.5.4]). In particular, finite extensions of $\mathbb{Q}(t_1, \ldots, t_r)$ and $\mathbb{Q}_p(t_1, \ldots, t_r)$ are Kummer-faithful. On the other hand, many of the infinite algebraic extensions of $\mathbb{Q}$ are not Kummer-faithful. For example, the field $\mathbb{Q}(\mu_{p^\infty})$ generated over $\mathbb{Q}$ by all p-power roots of unity is not Kummer-faithful. In general, it is not easy to determine whether a given field is Kummer-faithful or not. In this paper, we give in §2 a sufficient condition for a *Galois* extension of a finite number field to be Kummer-faithful in terms of ramification theory (Theorem 2.12 and Corollary 2.16), and in 3 construct in §3 examples of Kummer-faithful fields of infinite degree over $\mathbb{Q}$ which are not sub-p-adic for any prime number p. Some similar examples have already been studied in Ohtani [Oh]; we give a more systematic treatment in this paper.

Furthermore, we define a closely related notion "highly Kummer-faithful field" (Definition 2.6) and study its relation to Kummer-faithfulness. High Kummer-faithfulness is defined by the vanishing of the coinvariant spaces of the étale cohomology groups (with Tate twists) of proper smooth varieties over the field. Precisely speaking, we say that a perfect field K of characteristic p_K is highly Kummer-faithful if, for every finite extension L of K and every proper smooth variety X over L, it holds that

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(r))_{G_L} = 0 \quad \text{for any prime number } \ell \neq p_K \text{ and any } i, r \text{ with } i \neq 2r.$$

Although not a priori from the definition, it follows that, under a certain condition, high Kummer-faithfulness implies Kummer-faithfulness (Proposition 2.9). The above mentioned sufficient con-

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dition in §2 applies in fact also to high Kummer-faithfulness, and our examples of fields in §3 are highly Kummer-faithful.

Notation. A *number field* is a finite extension of the field $\mathbb{Q}$ of rational numbers. A *p-adic field* is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. For any field F, we fix a separable closure $\overline{F}$ of F and we denote by G_F the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F. If a group G acts on a vector space H, we denote by H^G (resp. H_G) the invariants (resp. coinvariants) of H by G; thus H^G (resp. H_G) is the maximal subspace (resp. quotient space) of H on which G acts trivially.

2 Highly Kummer-faithful fields

In this section, we recall the definition of Kummer-faithful fields (cf. [Mo, Def. 1.5], [Ho, Def. 1.2]) and define a notion of highly Kummer-faithful fields. We also study the relationship between Kummer-faithfulness, high Kummer-faithfulness and some other properties.

2.1 Definitions

Definition 2.1. Let M be a $\mathbb{Z}$-module and ℓ a prime number. We say that $P \in M$ is *divisible* (resp. *ℓ-divisible*) if, for any integer $n > 0$, there exists $Q \in M$ such that $P = nQ$ (resp. $P = \ell^n Q$). We denote by M_{div} (resp. $M_{\ell\text{-div}}$) the set of divisible (resp. ℓ-divisible) elements of M, that is,

$$M_{\mathrm{div}} = \bigcap_{n>0} nM, \qquad M_{\ell\text{-div}} = \bigcap_{n>0} \ell^n M.$$

Note that $P \in M$ is divisible if and only if it is ℓ-divisible for all primes ℓ; we have $M_{\mathrm{div}} = \bigcap_\ell M_{\ell\text{-div}}$.

Definition 2.2. A perfect field K is *Kummer-faithful* if, for every finite extension L of K and every semi-abelian variety A over L, it holds that

$$A(L)_{\mathrm{div}} = 0.$$

The facts below immediately follow from the definition of Kummer-faithfulness.

(1) If a perfect field K is Kummer-faithful, then so is any subfield of K.

(2) Let K' be a finite extension of a perfect field K. Then K is Kummer-faithful if and only if K' is Kummer-faithful.

Note that a perfect field K is Kummer-faithful if and only if $A(K)_{\mathrm{div}} = 0$ for every semi-abelian variety A over K (cf. [Mo, Rem. 1.5.2]).

It is also shown in [Mo, Rem. 1.5.4] that any sub-p-adic field is Kummer-faithful. Here, a field k is *sub-p-adic* if there exists a prime number p and a finitely generated field extension L of $\mathbb{Q}_p$ such that k is isomorphic to a subfield of L. In particular, a number field is Kummer-faithful.

Proposition 2.3. *A perfect field K is Kummer-faithful if and only if $\mathbb{G}_{\mathrm{m}}(L)_{\mathrm{div}} = 0$ for any finite extension L of K and $A(K)_{\mathrm{div}} = 0$ for any abelian variety A over K.*

Proof. It suffices to show the "if" direction. Let B be a semi-abelian variety over K; thus B is an extension of an abelian variety A by a torus T. It is enough to show $B(K)_{\mathrm{div}} = 0$ (see just after Definition 2.2). Assume that there exists a non-zero divisible element $P \in B(K)_{\mathrm{div}}$. For any integer $n > 0$, let X_n be the set of $P' \in B(K)$ such that $nP' = P$. The set X_n is a non-empty finite set. Furthermore, $\{X_n\}_n$ forms a projective system with transition maps $f_{n,m} \colon X_m \to X_n$ given by $f_{n,m}(P') = (m/n)P'$ for $n \mid m$. Then the projective limit $\varprojlim_n X_n$ is non-empty by [Bo, Chap. 1, §9.6, Prop. 8]. Take any element $(P_n)_n \in \varprojlim_n X_n$. Then the image $\bar{P}_n$ of P_n in $A(K)$

is divisible. Since $A(K)_{\rm div} = 0$ by assumption, $\bar{P}_n$ is zero for any n. Therefore, P_1 is a non-zero divisible element of $T(K)$. But this is a contradiction, since $T(K)_{\rm div} = 0$. Indeed, if the torus T splits by a finite extension L/K, then $T(K)$ is identified a subgroup of $\mathbb{G}_{\rm m}(L)^{\oplus d}$ with $d = \dim(T)$ and hence $T(K)_{\rm div} = 0$. □

For a semi-abelian variety A over a field k and a prime number ℓ, we set $T_\ell(A) := \varprojlim_n A[\ell^n]$ and $V_\ell(A) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(A)$, where the transition map $A[\ell^{n+1}] \to A[\ell^n]$ is given by the multiplication-by-ℓ. We call $T_\ell(A)$ and $V_\ell(A)$ respectively the *ℓ-adic* and *rational ℓ-adic Tate modules* of A. It is well-known that $T_\ell(A)$ is a free $\mathbb{Z}_\ell$-module of finite rank and $V_\ell(A)$ is a finite dimensional $\mathbb{Q}_\ell$-vector space. The Galois group G_k acts on $T_\ell(A)$ and $V_\ell(A)$ continuously.

Proposition 2.4. *Let A be a semi-abelian variety over a field k and K an algebraic extension of k.*
(1) *For any prime number ℓ, the following conditions are equivalent.*

(i) $(A(K)[\ell^\infty])_{\ell\text{-div}}$ *is zero.*

(ii) $A(K)[\ell^\infty]$ *is finite.*

(iii) $V_\ell(A)^{G_K} = 0$.

(2) *Consider the following conditions.*

(a) $A(K)_{\rm div}$ *is zero.*

(b) $(A(K)_{\rm tor})_{\rm div}$ *is zero.*

(c) $A(K)[\ell^\infty]$ *is finite for any prime number ℓ.*

Then we have (a) ⇒ (b) ⇔ (c). If k is Kummer-faithful and the extension K/k is Galois, then we have (a) ⇔ (b) ⇔ (c).

Remark 2.5. For the implication (b) ⇒ (a), the assumption that K/k be Galois is essential, as the following example shows: let $k = \mathbb{Q}$ and fix an integer $a > 1$. Let K be the extension of k obtained by adjoining the real nth roots of a for all integers $n \geq 1$. Then $\mathbb{G}_{\rm m}(K)_{\rm div} \neq 0$, whereas $(\mathbb{G}_{\rm m}(K)_{\rm tor})_{\rm div} = 0$.

Proof. (1) The equivalence of (ii) and (iii) follows immediately from the definition of Tate modules. We also have a natural isomorphism $T_\ell(A)^{G_K} \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/\ell]/\mathbb{Z}, A(K)[\ell^\infty])$, which implies the equivalence of (i) and (iii).
(2) The implication $(a) \Rightarrow (b)$ is clear. The implication $(b) \Rightarrow (c)$ follows from the natural isomorphism $\prod_\ell T_\ell(A)^{G_K} \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A(K)_{\rm tor})$, where ℓ ranges over the prime numbers. To show $(c) \Rightarrow (b)$, assume that a non-zero divisible element P of $A(K)_{\rm tor}$ exists. Let N be the order of P and take a prime divisor ℓ of N. Then we see that $(\ell^{-1}N)P$ is a non-zero ℓ-divisible element of $A(K)[\ell^\infty]$. By (1), we have that $A(K)[\ell^\infty]$ is infinite. This shows $(c) \Rightarrow (b)$.

Now we show $(b) \Rightarrow (a)$ under the assumption that k is Kummer-faithful and K/k is Galois. Assume that there exists a non-zero divisible element P of $A(K)$. For any integer $n > 0$, let X_n be the set of $P' \in A(K)$ such that $P = nP'$. As is explained in the proof of Proposition 2.3, we know that $\{X_n\}_n$ forms a projective system and its projective limit $\varprojlim_n X_n$ is non-empty. Take an element $(P_n)_n$ of $\varprojlim_n X_n$. Let k'/k be a finite subextension of K/k such that $P \in A(k')$. Since k is Kummer-faithful, k' is also Kummer-faithful. Hence we have $P_{n_0} \notin A(k')$ for some n_0. Take $\sigma_0 \in G_{k'}$ such that $\sigma_0(P_{n_0}) - P_{n_0} \neq 0$ and set $Q_n := \sigma_0(P_n) - P_n$ for $n > 0$. Then $Q_{n_0} \neq 0$ and $nQ_{nn_0} = Q_{n_0}$ for any n. Since the extension K/k is Galois, we have $Q_n \in A(K)[n]$ for any n. Hence Q_{n_0} is a non-zero divisible element of $A(K)_{\rm tor}$, which shows $(b) \Rightarrow (a)$. □

Definition 2.6. Let K be a perfect field with characteristic $p_K \geq 0$.
(1) We say that K is *quasi-highly Kummer-faithful* if, for every finite extension L of K and every proper smooth variety X over L, it holds that

$$(*) \quad H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)_{G_L} = 0 \quad \text{for any prime number } \ell \neq p_K \text{ and any } 0 < i \leq 2\dim X.$$

(2) We say that K is *highly Kummer-faithful* if, for every finite extension L of K and every proper smooth variety X over L, it holds that

$$(\sharp) \quad H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(r))_{G_L} = 0 \quad \text{for any prime number } \ell \neq p_K \text{ and any } i, r \text{ with } i \neq 2r.$$

We have the following implication:

$$K \text{ is highly Kummer-faithful} \Rightarrow K \text{ is quasi-highly Kummer-faithful.}$$

Note that the condition $(\sharp)$ is equivalent to the condition

$$(\sharp)' \quad H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(r))^{G_L} = 0 \quad \text{where } \ell \neq p_K \text{ is any prime number and } i \neq 2r.$$

Indeed, we have isomorphisms $(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(r))_{G_L})^\vee \simeq (H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(r))^\vee)^{G_L} \simeq H^{2g-i}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(g-r))^{G_L}$, where $g = \dim X$.

We also note that the facts below immediately follow from the definition of high Kummer-faithfulness (resp. quasi-high Kummer-faithfulness).

(1) If K is highly Kummer-faithful (resp. quasi-highly Kummer-faithful), then so is any subfield of K.

(2) Let K' be a finite extension of a perfect field K. Then K is highly Kummer-faithful (resp. quasi-highly Kummer-faithful) if and only if K' is highly Kummer-faithful (resp. quasi-highly Kummer-faithful).

Remark 2.7. In view of the fact that Kummer theory deals with abelian extensions of degree prime to the characteristic of the base field, we impose the condition $\ell \neq p_K$ in the above definition of (quasi-)high Kummer-faithfulness, though there may be room for discussion whether we should really do so.

Remark 2.8. We should note that Kummer-faithfulness and (quasi-)high Kummer-faithfulness are "local notion" in coefficients. In fact, it is not difficult to check that a perfect field K is Kummer-faithful if and only if K is *ℓ-Kummer-faithful* for any prime number ℓ. Here, K is ℓ-Kummer-faithful if, for every finite extension L of K and every semi-abelian variety A over L, it holds that $A(L)_{\ell\text{-div}} = 0$. If we define the notion of *high ℓ-Kummer-faithfulness* by the obvious manner, then K is highly Kummer-faithful if and only if K is highly ℓ-Kummer-faithful for any prime number ℓ.

Proposition 2.9. *Let K be a Galois extension of a Kummer-faithful field. If K is quasi-highly Kummer-faithful, then K is Kummer-faithful.*

Proof. Note that, for any finite extension L of K, any prime number ℓ and any g-dimensional abelian variety A over L, there exist G_L-equivalent isomorphisms $\mathbb{Q}_\ell(-1) \simeq H^2_{\text{ét}}(\mathbb{P}^1_{\overline{K}}, \mathbb{Q}_\ell)$ and $V_\ell(A) \simeq (H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell))^\vee$. Hence the result follows from Propositions 2.4 and 2.3. □

Proposition 2.10. *If A is a semi-abelian variety over a sub p-adic field K, then $A(K)_{\text{tor}}$ is finite.*

Proof. We may assume that K is a finite extension of $K_0 = \mathbb{Q}_p(t_1, \ldots, t_r)$ where $t_1, .., t_r$ are transcendental basis of K over $\mathbb{Q}_p$. The torsion part of K-rational points of any torus over K is finite. Thus it suffices to prove the proposition in the case where A is an abelian variety. Let $A_0 := \text{Res}_{K/K_0}(A)$ be the Weil restriction. Then A_0 is an abelian variety over K_0 and we have $A_0(K_0) = A(K)$. By the Lang-Néron theorem ([Co], [LN]), there exist an abelian variety A'_0 over $\mathbb{Q}_p$ and an injection $\iota\colon A'_0(\mathbb{Q}_p) \hookrightarrow A_0(K_0)$ with the property that $A_0(K_0)/\iota(A'_0(\mathbb{Q}_p))$ is a finitely generated abelian group. Since $A'_0(\mathbb{Q}_p)_{\text{tor}}$ is finite (cf. [Ma]), we obtain that $A_0(K_0)_{\text{tor}} = A(K)_{\text{tor}}$ is finite. □

Summary. Let K be a Galois extension of a Kummer-faithful field. For such K, we consider the following conditions:

(Sub$_p$) K is sub p-adic.

(Tor)$_{\text{fin.}}$ For any finite extension L of K and any semi-abelian variety A over L, it holds that $A(L)_{\text{tor}}$ is finite.

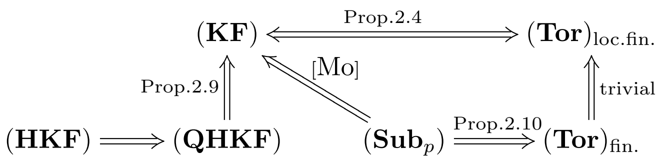
(Tor)$_{\text{loc.fin.}}$ For any finite extension L of K, any semi-abelian variety A over L and any prime number ℓ, it holds that $A(L)[\ell^\infty]$ is finite.

(KF) K is Kummer-faithful.

(QHKF) K is quasi-highly Kummer-faithful.

(HKF) K is highly Kummer-faithful.

Then we have the following diagram of logical relations:



It seems natural to consider the relationship between sub p-adic fields and (quasi-)highly Kummer-faithful fields. First, we note that *any sub p-adic field is not highly Kummer faithful.* To see this, it is enough to show that $\mathbb{Q}_p$ is not highly Kummer faithful. Let E be a Tate curve over $\mathbb{Q}_p$. Since we have a natural exact sequence $0 \to \mathbb{Q}_p(1) \to V_p(E) \to \mathbb{Q}_p \to 0$ of representations of $G_{\mathbb{Q}_p}$, we obtain $(H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p(1))_{G_{\mathbb{Q}_p}})^\vee \simeq (H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p(1))^\vee)^{G_{\mathbb{Q}_p}} \simeq V_p(E)(-1)^{G_{\mathbb{Q}_p}} \neq 0$. Thus we are done. Now we can ask the following weaker question:

Question. Are sub p-adic fields quasi-highly Kummer-faithful?

Under the "ℓ-adic" and "p-adic" monodromy weight conjectures (Conj. 4.1 and 5.1 of [Ja]), we can show the following.

Proposition 2.11. *Assume the monodromy weight conjectures. Then any p-adic field is quasi-highly Kummer-faithful.*

Proof. Let K be a p-adic field, L a finite extension of K and X a proper smooth variety over L of dimension g. Let ℓ be any prime number (including the case $\ell = p$). Since we have $(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)_{G_L})^\vee \simeq (H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^\vee)^{G_L} \simeq H^{2g-i}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(g))^{G_L}$, for our purpose, it suffices to check the vanishing of $H^{2g-i}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(g))^{G_L}$. Since we have $g \notin [\max\{0, g-i\}, (2g-i)/2]$ for $i > 0$, it follows from Corollaries 4.3 and 5.2 of [Ja] (under the assumption of the monodromy weight conjectures) that the above subspace is zero for any prime number ℓ. □

2.2 Criteria for high Kummer-faithfulness

We give some criteria for high Kummer-faithfulness in terms of ramification theory.

Theorem 2.12. *Let K be a Galois extension of a number field k. Assume that, for any finite extension k'/k, the ramification index of the maximal abelian subextension of Kk'/k' at any finite place of k' is finite. Then K is highly Kummer-faithful.*

In particular, any number field is highly Kummer faithful.

The theorem is a consequence of a general property of Galois representations (Lemma 2.14 below). To state it, we recall the notion of Weil weights.

Definition 2.13. Let k be a number field, v a finite place of k not above ℓ, V an ℓ-adic representation of G_k and w a real number. We say that V has (pure) *Weil weight w at v* if

(a) V is unramified at v,

(b) coefficients of the characteristic polynomial $\det(T - \mathrm{Frob}_v \mid V)$ are algebraic numbers, and

(c) for any root α of $\det(T - \mathrm{Frob}_v \mid V)$ and any embedding $\iota\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, it holds that $|\iota(\alpha)| = q_v^{w/2}$. Here, q_v is the order of the residue field of k at v.

For a proper smooth variety X over a number field k which has good reduction at a finite place v of k, the Weil conjecture proved by Deligne [De1, De2] implies that $H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(r))$ has Weil weight $i - 2r$ at v.

Lemma 2.14. *Let K be a Galois extension of a number field k and ℓ a prime number. Let M/k be the maximal abelian subextension of K/k. Assume that the ramification index of the extension M/k at any finite place of k above ℓ is finite. Then, for any $\mathbb{Q}_\ell$-representation V of G_k with non-zero Weil weight at some finite place of k not above ℓ, we have $V^{G_K} = 0$.*

Proof. Assume that $V^{G_K} \neq 0$. Then V^{G_K} is a non-zero G_k-stable submodule of V since K is a Galois extension of k. Let $\psi\colon G_k \to \mathbb{Q}_\ell^\times$ be the character such that $\det V^{G_K} = \mathbb{Q}_\ell(\psi)$. We denote by $k(\psi)/k$ the splitting field extension of ψ. It follows from class field theory that the extension $k(\psi)/k$ is potentially unramified at all finite places of k not above ℓ and is unramified at all but finitely many places of k. Since $k(\psi)$ is an abelian subextension of K/k, we have $k(\psi) \subset M$. Hence the assumption on the extension M/k implies that the ramification degree of $k(\psi)/k$ at any finite place of k above ℓ is finite. Then class field theory implies that the extension $k(\psi)/k$ is finite, and so $\psi(G_k)$ is finite. This contradicts the assumption that V (and hence $\det V = \mathbb{Q}_\ell(\psi)$ also) has non-zero Weil weight at some finite place of k. □

Proof of Theorem 2.12. Let L be a finite extension of K, X a proper smooth variety over L, ℓ a prime number and i, r be integers such that $i \neq 2r$. Put $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(r))$. We take a finite extension k' of k so that $L = Kk'$ and X is defined over k'. Then, V is a $\mathbb{Q}_\ell$-representation of $G_{k'}$ with non-zero Weil weight $i - 2r$ at any finite place v of k not above ℓ such that X has good reduction at v. It follows from the assumption on K, k and Lemma 2.14, that $V^{G_L} = 0$. □

Definition 2.15. (1) Let K be an algebraic extension of a p-adic field k and $\tilde{K}$ the Galois closure of K/k. Put $G = \mathrm{Gal}(\tilde{K}/k)$. We say that the extension K/k has *finite maximal ramification break* if G^c is trivial[1] for c large enough. In the case where K is an abelian extension of k, this is equivalent to the condition that the ramification index of K/k is finite by local class field theory.
(2) Let K be a Galois extension of a number field k. For any finite place v of k, We say that the extension K/k has a *finite maximal ramification break at v* if the extension K_w/k_v has finite maximal ramification break where w is a finite place of K above v. Note that this definition does not depend on the choice of w. Note also that, if K is an abelian extension of a number field k, K/k has finite maximal ramification break at a finite place v of k if and only if the ramification index of K/k at v is finite.
(3) Let K be a Galois extension of a number field k. We say that the extension K/k has *finite maximal ramification break everywhere* if it has finite maximal ramification break at any finite place v of k.

Note that the property "having finite maximal ramification break" is unchanged by finite extension of the base field k. Hence the following corollary follows from Theorem 2.12:

Corollary 2.16. *Let K be a Galois extension of a number field k. Assume that the extension K/k has finite maximal ramification break everywhere. Then K is highly Kummer-faithful.*

[1] We follow Serre's convention ([Se1], Chap. IV, §3) for the i-th upper ramification subgroup G^i. In particular, $G^{-1} = G$, G^i for $-1 < i \leq 0$ is the inertia subgroup of G, and $G^{0+} := \bigcup_{i>0} G^i$ is the wild inertia subgroup of G.

3 Examples

In this section, we give some examples of highly Kummer-faithful fields which are infinite algebraic extensions of number fields. To check high Kummer-faithfulness, we often use Theorem 2.12, Corollary 2.16 and the following lemma:

Lemma 3.1. *Let k be a p-adic field and $\{K_i\}_{i\in I}$ a family of Galois extensions of k. Let K_I be the composite field of the K_i's for all $i \in I$. Then, for a real number $c \geq -1$, $\mathrm{Gal}(K_I/k)^c$ is trivial if and only if $\mathrm{Gal}(K_i/k)^c$ is trivial for any $i \in I$.*

Proof. By [Se1, Chap. IV, Prop. 14], the natural surjection $\mathrm{Gal}(K_I/k) \twoheadrightarrow \mathrm{Gal}(K_i/k)$ induces a surjection $\mathrm{Gal}(K_I/k)^c \twoheadrightarrow \mathrm{Gal}(K_i/k)^c$. Furthermore, we have a natural injection $\mathrm{Gal}(K_I/k) \hookrightarrow \prod_{i\in I} \mathrm{Gal}(K_i/k)$. The lemma immediately follows from these maps. □

The following is an immediate consequence of Corollary 2.16 and Lemma 3.1.

Corollary 3.2. *Let k be a number field.*
(1) *Let $d > 0$ be an integer. Then the composite field of all finite extensions of k of degree $\leq d$ is highly Kummer faithful.*
(2) *Let $\{k_i\}_{i\in I}$ be a family of finite extensions of k. Assume that the discriminants of k_i/k are prime to each other. Then the composite field of k_i's for all $i \in I$ is highly Kummer faithful.*

3.1 Construction of highly Kummer-faithful fields from semi-abelian varieties

Let k be a number field. Let $g > 0$ be an integer and $\mathbf{m} := (m_p)_p$ a family of non-negative integers where p ranges over the prime numbers. Let $k_{g,\mathbf{m}}$ be the extension field of k obtained by adjoining all coordinates of elements of $B[p^{m_p}]$ for all semi-abelian varieties B over k of dimension at most g and all prime numbers p.

Theorem 3.3. (1) *The extension $k_{g,\mathbf{m}}/k$ has finite maximal ramification break everywhere.*
(2) *The field $k_{g,\mathbf{m}}$ is highly Kummer-faithful.*

Proof. (2) follows from (1) by Corollary 2.16. To prove (1), take any finite place v of k. By Lemma 3.1, it suffices to show that there exists an integer $c_v > 0$ with the property that $\mathrm{Gal}(k_v(B[p^{m_p}])/k_v)^{c_v}$ is trivial for any semi-abelian variety B over k of dimension at most g and any prime number p.

Let B_{v/k_v} be a semi-abelian variety of dimension at most g which is an extension of an abelian variety A_{v/k_v} by a torus T_{v/k_v}. By Raynaud's criterion of semi-stable reduction [Gr, Prop. 4.7], the abelian variety A_v has semi-stable reduction over $k_v(A_v[12])$. We clearly have $[k_v(A_v[12]) : k_v] \leq \#\mathrm{GL}_{2g}(\mathbb{Z}/12\mathbb{Z}) =: c(g)$. On the other hand, set $X(T) := \mathrm{Hom}_{\overline{k}_v}(T_v, \mathbf{G}_m)$ and $r := \dim T_v$. Then $X(T)$ is a free $\mathbb{Z}$-module of rank r and G_{k_v} acts on $X(T)$. If we denote by $k_{v,0}/k_v$ the splitting field of this G_{k_v} action on $X(T)$, then the extension $k_{v,0}/k_v$ is finite and Galois, the torus T_v splits over $k_{v,0}$ and we have an inclusion $\mathrm{Gal}(k_{v,0}/k_v) \hookrightarrow \mathrm{GL}_r(\mathbb{Z})$. It is shown by Minkowski[2] [Mi] that there exists an integer $c'(g) > 0$, depending only on g, such that the order of any torsion subgroup of $\mathrm{GL}_g(\mathbb{Z})$ is bounded by $c'(g)$. Thus we have $[k_{v,0} : k_v] \leq c'(g)$. Now we denote by k'_v the composite field of all Galois extensions of k_v of degree at most $\max\{c(g), c'(g)\}$. Since there exist only finitely many finite extensions of given degree over a p-adic field, we see that k'_v is a finite Galois extension of k_v. We note that it follows from the above observation that, for any semi-abelian variety B_{v/k_v} of dimension at most g, $B_v \otimes_{k_v} k'_v$ is an extension of an abelian variety with semi-stable reduction by a split torus. Take a real number $c'_v > 0$ so that $\mathrm{Gal}(k'_v/k_v)^{c'_v}$ is trivial.

Let B_{v/k_v} be a semi-abelian variety of dimension at most g. If we denote by p_v the prime number below v, then we have $[k_v(B[p_v^{m_{p_v}}]) : k_v] \leq \#\mathrm{GL}_{2g}(\mathbb{Z}/p_v^{m_{p_v}}\mathbb{Z}) =: c(g, \mathbf{m}, v)$. We denote

[2] More easily, the existence of $c'(g)$ immediately follows from the fact that the kernel of the projection $\mathrm{GL}_r(\mathbb{Z}) \to \mathrm{GL}_r(\mathbb{Z}/3\mathbb{Z})$ is torsion free.

by k_v'' the composite field of all Galois extensions of k_v of degree at most $c(g, \mathbf{m}, v)$. Take a real number $c_v'' > 0$ so that $\mathrm{Gal}(k_v''/k_v)^{c_v''}$ is trivial.

Now we set $c_v := \max\{c_v', c_v''\}$. We show that this c_v has the desired property. Let $B_{/k}$ be a semi-abelian variety of dimension at most g which is an extension of an abelian variety $A_{/k}$ by a torus $T_{/k}$. For the prime p below v, the inequality $c_v \geq c_v''$ implies that $\mathrm{Gal}(k_v(B[p^{m_p}])/k_v)^{c_v}$ is trivial. Next we take any prime p which is not below v. Let k_v' be the finite Galois extension of k_v defined above. Since $T \otimes_k k_v'$ is a split torus and $A \otimes_k k_v'$ has semi-stable reduction, we see that the inertia subgroup I_v' of $G_{k_v'}$ acts unipotently on $V_p(B)$. In particular, $\rho_{B,p}(I_v')$ is a pro-p group where $\rho_{B,p} \colon G_k \to \mathrm{GL}_{\mathbb{Q}_p}(V_p(B))$ is the continuous homomorphism obtained by the G_k-action on $V_p(B)$. On the other hand, we have $G_{k_v}^{c_v} \subset G_{k_v}^{c_v'} \subset I_v'$ by definition of c_v and c_v', and $G_{k_v}^{c_v'}$ is a pro-p_v group since $c_v' > 0$. Hence $\rho_{B,p}(G_{k_v}^{c_v})$ is trivial, which implies that $\mathrm{Gal}(k_v(B[p^{m_p}])/k_v)^{c_v}$ is trivial as desired. □

Remark 3.4. We recall that any sub-p-adic field is Kummer-faithful (cf. [Mo, Rem. 1.5.4]). If $m_p > 0$ for infinitely many p, our field $k_{g,\mathbf{m}}$ above gives an example of a field that is not sub-p-adic but Kummer-faithful. Indeed, let M be the subfield of $k_{g,\mathbf{m}}$ obtained by adjoining to $\mathbb{Q}$ all pth roots of unity for all primes p such that $m_p > 0$. If $k_{g,\mathbf{m}}$ is sub-p-adic, then M is also sub-p-adic. However, this is impossible since the residue field of M at any finite place is infinite.

3.2 High Kummer-faithfulness of abelian extensions over $\mathbb{Q}$

Theorem 2.12 says that the condition of finite ramification is sufficient for a Galois extension K of a number field k to be highly Kummer-faithful. (Thus, for example, if K/k is a class field tower, then K is highly Kummer-faithful.) It is natural to ask if it is also necessary. Unfortunately, the necessity does not hold in general; see §3.3. If, however, K is an *abelian extension of* $\mathbb{Q}$, then Proposition 3.5 below shows the necessity.

Now we set $\mu_{(2)} := \mu_4$, $\mu_{(\ell)} := \mu_\ell$ for odd primes ℓ and set $\mu := \bigcup_\ell \mu_{(\ell)}$. Following [Mo, Def. 1.5], we say that a perfect field K is *torally Kummer-faithful* if, for every finite extension L of K and every torus T over L, it holds that

$$T(L)_{\mathrm{div}} = 0.$$

By definition, Kummer-faithful fields are torally Kummer faithful.

Proposition 3.5. *Let K be an abelian extension of $\mathbb{Q}$. Then the following are equivalent.*

(1) *The ramification index of the extension $K/\mathbb{Q}$ is finite at any prime number.*

(2) *K is highly Kummer-faithful.*

(3) *K is Kummer-faithful.*

(4) *K is torally Kummer-faithful.*

(5) *The ℓ-adic cyclotomic character $\chi_\ell : G_K \to \mathbb{Z}_\ell^\times$ has open image for any prime number ℓ.*

(6) *K is a subfield of $\mathbb{Q}(\mu_{p^{n_p}} \mid p$: prime$)$ for some family of positive integers $(n_p)_p$.*

Proof. We know already that (1) ⇒ (2) ⇒ (3) ⇒ (4) (see Corollary 2.16, Proposition 2.9). The implication (4) ⇒ (5) is easy to see, as noted in Remark 1.5.1 of [Mo]. The equivalence of (1) and (6) is clear from class field theory. It remains to prove (5) ⇒ (1). Let $K(\mu_{\ell^\infty})$ be the maximal ℓ-power cyclotomic extension of K. Since $K/\mathbb{Q}$ is abelian, by class field theory, the inertia group of $K(\mu_{\ell^\infty})/\mathbb{Q}$ at ℓ is isomorphic to $\mathbb{Z}_\ell^\times$, and that of $K(\mu_{\ell^\infty})/K$ at a place above ℓ is identified with an open subgroup of $\mathrm{Im}(\chi_\ell)$. Thus if $\mathrm{Im}(\chi_\ell)$ is open in $\mathbb{Z}_\ell^\times$, the ramification index of $K/\mathbb{Q}$ at ℓ is finite. □

3.3 Kummer-faithful fields with infinite ramification

The finite ramification condition is in general not necessary for a Galois extension of a number field to be Kummer-faithful. In this subsection, we apply the main theorem of [Oz] to produce Kummer-faithful fields with infinite ramification.

Theorem 3.6 ([Oz, Theorem 1.1])**.** *Let k be a Galois extension of $\mathbb{Q}_p$ of degree d. Let π be a uniformizer of k and q the order of the residue field of k. Denote by k_π/k the Lubin-Tate extension associated with π. Assume that neither of the following two conditions hold.*

(a) $q^{-1}N_{k/\mathbb{Q}_p}(\pi)$ *is a root of unity, where* $N_{k/\mathbb{Q}_p}$ *denotes the norm map of* $k/\mathbb{Q}_p$.

(b) $N_{k/\mathbb{Q}_p}(\pi)$ *is a q-Weil integer of weight d/t for some integer $1 \leq t \leq d$.*

Then k_π is Kummer-faithful.

Note that the theorem is proved in a more general setting in [Oz]. However, we content ourselves with the above version to keep the notation simple in the following construction of Kummer-faithful fields,

Now let k be a number field. Let p be a prime number which splits completely in k, so that we identify the completion of k at a place above p with $\mathbb{Q}_p$. Let π be an element of k which gives rise to a uniformizer of $\mathbb{Q}_p$. Set $f(x) := \pi x + x^p \in k[x]$. Let K be the extension field of k obtained by adjoining all roots of $f^n(x) = 0$ for all $n > 0$ (here f^n is the n-th composite of f). We apply the theorem with $d = 1$; if the complex absolute value of π with respect to some embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ is not equal to either p or $\sqrt{p}$, then K is Kummer-faithful. Indeed, K is a subfield of the Lubin-Tate extension k_π of $\mathbb{Q}_p$ defined by $f(x)$ and Theorem 3.6 shows that k_π is Kummer-faithful.

3.4 A remark on the composite of Kummer-faithful fields

The condition "having finite maximal ramification break" is closed under the composition of fields (Lemma 3.1). In view of Corollary 2.16, one may wonder if the composite field of Kummer-faithful Galois extensions of a number field is again Kummer-faithful. At present, we do not know if this is true or not. In the local case, however, this is *not* true, as the following example shows:

Example 3.7. For uniformizers π_1 and π_2 of $(k =)\mathbb{Q}_p$, we consider the following properties.

(1) The Lubin-Tate extensions k_{π_1} and k_{π_2} of $k := \mathbb{Q}_p$ associated with π_1 and π_2 respectively, are Kummer-faithful.

(2) The intersection $k_{\pi_1} \cap k_{\pi_2}$ is a finite extension of $\mathbb{Q}_p$.

We claim that, if π_1 and π_2 satisfy the condition (2), then the composite field K of k_{π_1} and k_{π_2} is not Kummer-faithful. Indeed, let M be the maximal abelian pro-p extension of $\mathbb{Q}_p$. By the condition (2), the maximal pro-p subextension $L/\mathbb{Q}_p$ of $K/\mathbb{Q}_p$ is a $\mathbb{Z}_p^2$-extension and the extension K/L is finite. In particular, it follows from local class field theory that M is a finite extension of L. On the other hand, M is not Kummer-faithful since $M(\mu_p)$ contains all p-power roots of unity. Then it follows that L and K are not Kummer-faithful.

Now we show that there exists a pair (π_1, π_2) which satisfies both conditions (1) and (2). We first note that, by Theorem 3.6, the condition (1) holds if $p^{-1}\pi_i$ is not a root of unity and π_i is not a p-Weil integer of weight 1 for $i = 1, 2$. Hence, to show the existence of a desired pair (π_1, π_2), it is enough to prove that the condition (2) holds if $\pi_1\pi_2^{-1}$ is not a root of unity. Assume that $k_{\pi_1} \cap k_{\pi_2}$ is an infinite extension of $\mathbb{Q}_p$. Let $k_{\pi_i}^1$ be the subextension of $k_{\pi_i}/\mathbb{Q}_p$ of degree $p-1$ and set $K_1 := k_{\pi_1}k_{\pi_2}^1$ and $K_2 := k_{\pi_2}k_{\pi_1}^1$. Then extensions $K_1/k_{\pi_1}^1 k_{\pi_2}^1$ and $K_2/k_{\pi_1}^1 k_{\pi_2}^1$ are $\mathbb{Z}_p$-extensions. By the assumption, we have that $K_1 \cap K_2$ is an infinite degree extension of $k_{\pi_1}^1 k_{\pi_2}^1$. This implies the equality $K_1 \cap K_2 = K_1 = K_2$. Now we denote by $\chi_{\pi_i} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ the Lubin-Tate character associated with π_i (cf. [Se2, Chap. III, A4]). If we regard χ_{π_i} as a continuous character $\mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ by local class field theory, then χ_{π_i} is characterized by the property that $\chi_{\pi_i}(\pi_i) = 1$

and $\chi_{\pi_i}(u) = u^{-1}$ for any $u \in \mathbb{Z}_p^\times$. Note that the extension field of $\mathbb{Q}_p$ which corresponds to the kernel of χ_{π_i} is k_{π_i}. We denote by $\eta = \chi_{\pi_1}\chi_{\pi_2}^{-1}$ and also denote by $\mathbb{Q}_p(\eta)$ the extension field of $\mathbb{Q}_p$ which corresponds to the kernel of η. Then $\mathbb{Q}_p(\eta)$ is an unramified subextension of $k_{\pi_1}k_{\pi_2}/\mathbb{Q}_p$, and it follows from the equality $k_{\pi_1}k_{\pi_2} = K_1K_2 = K_1$ that the residue field of $k_{\pi_1}k_{\pi_2}$ is finite. Hence we obtain the fact that $\mathbb{Q}_p(\eta)$ is a finite (unramified) extension of $\mathbb{Q}_p$, that is, $\eta(G_{\mathbb{Q}_p})$ is finite. Thus we have $(\chi_{\pi_1}\chi_{\pi_2}^{-1})^j = 1$ for some integer $j > 0$. Since we have $\chi_{\pi_1}(\pi_2)\chi_{\pi_2}^{-1}(\pi_2) = \chi_{\pi_1}(\pi_1 \cdot (\pi_1^{-1}\pi_2))\chi_{\pi_2}^{-1}(\pi_2) = \pi_1\pi_2^{-1}$, it follows that $\pi_1\pi_2^{-1}$ is a root of unity as desired.

3.5 Construction of highly Kummer-faithful fields via integral p-adic Hodge theory

Let k be a number field. Let $\mathcal{L} = (\mathcal{L}_v)_v$ be a collection of open normal subgroup $\mathcal{L}_v$ of I_v for each finite place v of k. Let V be a geometric ℓ-adic representation of G_k (in the sense of [FM]). We say that the *inertial level of V is bounded by $\mathcal{L}$* if $V|_{\mathcal{L}_v}$ is semi-stable at every v. Let $h \geq 0$ be an integer. We say that the *length of Hodge-Tate weights of V is bounded by h* if, for each v above ℓ, there exists integers $a_v \leq b_v$ such that $b_v - a_v \leq h$ and $\mathrm{HT}_v(V) \subset [a_v, b_v]$. Here, $\mathrm{HT}_v(V)$ is the set of Hodge-Tate weights of V at v. (We normalize the notion of Hodge-Tate weights so that the Hodge-Tate weight of cyclotomic character is one.) For a prime number ℓ, we denote by $\mathcal{R}^{\ell}_{\mathcal{L},h}(G_k)$ the set of geometric ℓ-adic representations of G_k whose inertial level are bounded by $\mathcal{L}$ and the length of Hodge-Tate weights are bounded by h.

Definition 3.8. Let T be a torsion $\mathbb{Z}_\ell$-representation of G_k. We say that T *comes from $\mathcal{R}^{\ell}_{\mathcal{L},h}(G_k)$* if there exist $V \in \mathcal{R}^{\ell}_{\mathcal{L},h}(G_k)$ and G_k-stable $\mathbb{Z}_\ell$-lattices $L \subset L'$ in V such that $T \simeq L'/L$.

Let $\mathbf{m} := (m_p)_p$ be a family of non-negative integers where p ranges over the prime numbers. We denote by $k_{\mathcal{L},h,\mathbf{m}}$ the extension field of k obtained by adjoining all splitting fields of all torsion $\mathbb{Z}_\ell$-representations T coming from $\mathcal{R}^{\ell}_{\mathcal{L},h}(G_k)$ with $\ell^{m_\ell}T = 0$ for all prime numbers ℓ.

Theorem 3.9. (1) *The extension $k_{\mathcal{L},h,\mathbf{m}}/k$ has finite maximal ramification break everywhere.*
(2) *The field $k_{\mathcal{L},h,\mathbf{m}}$ is highly Kummer-faithful.*

Proof. (2) follows from (1) by Corollary 2.16. To prove (1), we first reduce ourselves to the case where $\sqrt{-1} \in k$. Put $k' = k(\sqrt{-1})$. For any finite place v' of k', set $\mathcal{L}'_{v'} := G_{k'_{v'}} \cap \mathcal{L}_v$ where v is the finite place of k below v'. Then $\mathcal{L}'_{v'}$ is an open subgroup of $I_{v'}$. Put $\mathcal{L}' = (\mathcal{L}'_{v'})_{v'}$ where v' runs thorough all finite places of k'. For any $V \in \mathcal{R}^{\ell}_{\mathcal{L},h}(G_k)$, we see $V|_{G_{k'}} \in \mathcal{R}^{\ell}_{\mathcal{L}',h}(G_{k'})$. Thus we have inclusions $k_{\mathcal{L},h,\mathbf{m}} \subset k_{\mathcal{L},h,\mathbf{m}}k' \subset k'_{\mathcal{L}',h,\mathbf{m}}$. It follows from this that, for the proof, it is enough to show that $k'_{\mathcal{L}',h,\mathbf{m}}/k'$ has finite maximal ramification break everywhere.

In the rest of the proof, we assume $\sqrt{-1} \in k$. Take any finite place v of k and denote by p the prime number below v. We take a finite Galois extension k'_v/k_v such that the inertia subgroup $I_{k'_v}$ of k'_v is contained in $\mathcal{L}_v$. Let $c'_v > 0$ be a real number such that $\mathrm{Gal}(k'_v/k_v)^{c'_v}$ is trivial. Let T be any torsion $\mathbb{Z}_\ell$-representation coming from $\mathcal{R}^{\ell}_{\mathcal{L},h}(G_k)$ with $\ell^{m_\ell}T = 0$. Thus there exist $V \in \mathcal{R}^{\ell}_{\mathcal{L},h}(G_k)$ and G_k-stable $\mathbb{Z}_\ell$ lattices $L \subset L'$ in V such that $T \simeq L'/L$. We denote by $k_v(T)$ the completion of the splitting field $k(T)$ of the representation T of G_k at a place above v. For the proof, by Lemma 3.1, it suffices to show that there exists a real number $c_v > 0$, depending only on $\mathcal{L}, h, m_p$ and v (not on T), with the property that $\mathrm{Gal}(k_v(T)/k_v)^{c_v}$ is trivial.

First we consider the case $\ell \neq p$. Let $\rho\colon G_k \to GL_{\mathbb{Q}_\ell}(V)$ be the continuous homomorphism describing the G_k-action on V. It follows from the fact that $I_{k'_v}$-action on V is unipotent that the group $\rho(I_{k'_v})$ is pro-ℓ. Since the group $G_{k_v}^{c'_v}$ is pro-p and this is a subgroup of $I_{k'_v}$, we obtain that $\rho(G_{k_v}^{c'_v})$ is trivial. This in particular implies that $\mathrm{Gal}(k_v(T)/k_v)^{c_v}$ is trivial.

Next we consider the case $\ell = p$. Let $t > 0$ be an integer so that $\chi_p^t \bmod p^{m_p} = 1$ where $\chi_p\colon G_k \to \mathbb{Z}_p{}^\times$ is the p-adic cyclotomic character. Since we have $T = T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(nt)$ for any integer n, we may suppose that $\mathrm{HT}_v(V)$ is contained in $[0, r]$ where $r := t + h$. Then $T|_{G_{k'_v}}$ is a quotient of two $G_{k'_v}$-stable lattices in a semi-stable $\mathbb{Q}_p$-representation with Hodge-Tate weights in $[0, r]$. Since T is killed by p^{m_p}, it follows from Theorem 1.1 of [CL][3] that there exists an integer $c''_v > 0$, depending only on k'_v, m_p and r, such that $G_{k'_v}^{c''_v}$ acts on T trivially (see also Theorem 1.1 of [Ha]). Furthermore, we have

$$G_{k'_v}^{c''_v} = G_{k'_v} \cap G_{k_v}^{\varphi_{k'_v/k_v}(c''_v)} \supset G_{k_v}^{c'_v} \cap G_{k_v}^{\varphi_{k'_v/k_v}(c''_v)} = G_{k_v}^{c_v}$$

where $c_v := \max\{c'_v, \varphi_{k'_v/k_v}(c''_v)\}$ and $\varphi_{k'_v/k_v}(u)$ is the Herbrand function for k'_v/k_v, that is,

$$\varphi_{k'_v/k_v}(u) := \int_0^u \frac{1}{(\mathrm{Gal}(k'_v/k_v)_0 : \mathrm{Gal}(k'_v/k_v)_t)}\, dt.$$

Hence the action of $G_{k_v}^{c_v}$ on T is trivial. Therefore, we obtain that $\mathrm{Gal}(k_v(T)/k_v)^{c_v}$ is trivial. Note that we may say that c_v is determined by $\mathcal{L}, h, m_p$ and v.

Now the constant c_v satisfies the desired property. □

Let $d > 0$ be an integer and $\mathbf{m} := (m_p)_p$ a family of non-negative integers where p ranges over the prime numbers. Let $k_{d\text{-st},\mathbf{m}}$ be the extension field of k obtained by adjoining all coordinates of elements of $A[p^{m_p}]$ for all prime numbers p and all abelian varieties A over k which have semi-stable reduction everywhere over an extension of k of degree at most d. Note that we have no restriction on the dimensions of abelian varieties for this definition; this is a remarkable difference between definitions of $k_{d\text{-st},\mathbf{m}}$ and $k_{g,\mathbf{m}}$. As a consequence of Theorem 3.9, we obtain the following.

Corollary 3.10. (1) *The extension $k_{d\text{-st},\mathbf{m}}/k$ has finite maximal ramification break everywhere.*
(2) *The field $k_{d\text{-st},\mathbf{m}}$ is highly Kummer-faithful.*

Proof. For each finite place v of k, let k'_v be the composite field of all Galois extensions of k_v of degree at most d. Note that k'_v is a finite extension of k_v. Let $\mathcal{L}_v$ $(\subset I_v)$ be the inertia group of k'_v and set $\mathcal{L} := (\mathcal{L}_v)_v$. Then the result follows from the fact that $k_{d\text{-st},\mathbf{m}}$ is contained in $k_{\mathcal{L},h,\mathbf{m}}$ with $h = 1$. □

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[3] Note that [CL] studies integral p-adic Hodge theory for odd primes p. However, the arguments given in [CL] work also for the case $p = 2$ under the condition that $\hat{G} = G_{p^\infty} \rtimes H_K$, that is, $K_{p^\infty} \cap K_\infty = K$. This condition holds for any odd primes p. In the case $p = 2$, the condition does not hold in general but it holds if $\sqrt{-1} \in K$. For this, see Proposition 4.1.5 of [Li]. Our assumption $\sqrt{-1} \in k$ at the beginning of this proof is needed only here.

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